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BULGARIAN MATHEMATICAL COMPETITIONS

2003 - 2006

GIL

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PREFACE

Bulgaria is a country with long traditions in mathematical competitions. There are numerous regional competitions connected with important dates in Christian calendar or in Bulgarian history. These competitions range in format and difficulty and give opportunity to all students in lower and secondary school to test their abilities in problem solving. Great many of them being fascinated by problem solving in such competitions start working hard in order to acquire new knowledge in mathematics.

The most important and prestigious national competitions in Bulgaria are Winter Mathematical Competition, Spring Mathematical Competition and National Olympiad. The organization of these competitions is responsibility of the Ministry of Education and Science, the Union of Bulgarian Mathematicians and the local organizers. The problems for the competitions are prepared by so called Team for extra curricula research – a specialized body of the Union of Bulgarian Mathematicians.

Winter Mathematical Competition. The first Winter Mathematical Competition took place in year 1982 in town of Russe. Since then it is held every year at the end of January or the beginning of February and about 1000 students from grades 4 to 12 take part in it. Four Bulgarian towns Varna, Russe, Bourgas and Pleven in turn host the competition.

Spring Mathematical Competition. The first Spring Mathematical Competition took place in year 1971 in town of Kazanlyk. The competition is being held annually at the end of March. Every year about 500 students from grades 8 to 12 take part in the competition. Two Bulgarian cities, Kazanlyk and Iambol in turn host the competition. The competition in town of Iambol is named after Atanas Radev (1886 – 1970). He was a famous teacher in mathematics who at the time of his life contributed enormously to mathematics education.

The results from Winter Mathematical Competition and Spring Mathematical Tournament are taken into consideration for selecting the candidates for Bulgarian Balkan Mathematical Olympiad team. Two selection tests then determine the team.

National Olympiad. The first National Mathematical Olympiad dates back in 1949-1950 school year. Now it is organized in three rounds – school, regional and national. The school round is carried out in different grades and is organized by regional mathematical authorities. They work out the problems and grade the solutions. The regional round, which is also carried in different grades, is organized in regional centers and the problems are now given by National Olympiad Commission. The grading is responsibility of the regional mathematical authorities. The national round is set in two days for three problems each day. The problems and organization are similar to these of the International Mathematical Olympiad (IMO). The best 12 students are invited

to take part in two selection tests. As a rule, each selection test is executed in two days, three problems per day. The results of these tests determine the six students for Bulgarian IMO team.

Bulgaria and international competitions in mathematics. Bulgaria is among the six countries (Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Romania and Union of the Soviet Socialist Republic) that initiated in year 1959, now extremely popular, International Mathematical Olympiad. Since then Bulgarian team took part in all IMO's. Bulgarian students take part also in gaining popularity Balkan Mathematical Olympiad and in the final round of the All Russian Mathematical Olympiad.

This book contains all problems for grades 8 to 12 from the above mentioned national competitions in the period 2003–2006. The problems from all selection tests for BMO and IMO are also included. Most of the problems are regarded as difficult IMO type problems. The book is intended for undergraduates, high school students and teachers who are interested in olympiad mathematics.

Sofia, Bulgaria
May, 2007

The authors

Bulgarian Mathematical Competitions 2003

Winter Mathematical Competition

Varna, January 30 – February 1, 2003

Problem 9.1. Let ABC be an isosceles triangle with $AC = BC$ and let k be a circle with center C and radius less than the altitude CH, $H \in AB$. Lines through A and B are tangent to k at points P and Q lying on the same side of the line CH. Prove that the points P, Q and H are collinear.

Oleg Mushkarov

Problem 9.2. Find all values of a, for which the equation

$$\frac{2a}{(x+1)^2} + \frac{a+1}{x+1} - \frac{2(a+1)x-(a+3)}{2x^2-x-1} = 0$$

has two real roots x_1 and x_2 satisfying the relation $x_2^2 - ax_1 = a^2 - a - 1$.

Ivan Landjev

Problem 9.3. Find the number of positive integers a less than 2003, for which there exists a positive integer n such that 3^{2003} divides $n^3 + a$.

Emil Kolev, Nikolai Nikolov

Problem 10.1. Find all values of a, for which the equation

$$\sqrt{ax^2+ax+2} = ax+2$$

has a unique root.

Alexander Ivanov, Emil Kolev

Problem 10.2. Let k_1 and k_2 be circles with centers O_1 and O_2, $O_1O_2 = 25$, and radii $R_1 = 4$ and $R_2 = 16$, respectively. Consider a circle k such that k_1 is internally tangent to k at a point A, and k_2 is externally tangent to k at a point B.

a) Prove that the segment AB passes trough a constant point (i.e., independent on k).

b) The line O_1O_2 intersects k_1 and k_2 at points P and Q, respectively, such that O_1 lies on the segment PQ and O_2 does not. Prove that the points P, A, Q and B are concyclic.

c) Find the minimum possible length of the segmentAB (when k varies).

Stoyan Atanasov, Emil Kolev

Problem 10.3. Let A be the set of all 4-tuples of 0 and 1. Two such 4-tuples are called *neighbors* if they coincide exactly at three positions. Let M be a subset of A with the following property: any two elements of M are not neighbors and there exists an element of M which is neighbor of exactly one of them. Find the minimum possible cardinality of M.

Ivan Landjev, Emil Kolev

Problem 11.1. Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{2a_n}$ for $n \geq 1$. Prove that:

a) $n \leq a_n^2 < n + \sqrt[3]{n}$; b) $\lim\limits_{n\to\infty}(a_n - \sqrt{n}) = 0$.

Nikolai Nikolov

Problem 11.2. Let M be an interior point of $\triangle ABC$. The lines AM, BM and CM meet the lines BC, CA and AB at points A_1, B_1 and C_1, respectively, such that $S_{CB_1M} = 2S_{AC_1M}$. Prove that A_1 is the midpoint of the segment BC if and only if $S_{BA_1M} = 3S_{AC_1M}$.

Oleg Mushkarov

Problem 11.3. Aleksander writes a positive integer as a coefficient of a polynomial of degree four, then Elitza writes a positive integer as another coefficient of the same polynomial and so on till all the five coefficients of the polynomial are filled in. Aleksander wins if the polynomial obtained has an integer root; otherwise, Elitza wins. Who of them has a winning strategy?

Nikolai Nikolov

Problem 12.1. Consider the polynomial $f(x) = 4x^4 + 6x^3 + 2x^2 + 2003x - 2003^2$. Prove that:

a) the local extrema of $f'(x)$ are positive;

b) the equation $f(x) = 0$ has exactly two real roots and find them.

Sava Grozdev, Svetlozar Doychev

Problem 12.2. Let M, N and P be points on the sides AB, BC and CA of $\triangle ABC$, respectively. The lines through M, N and P, parallel to BC, AC and AB, respectively, meet at a point T. Prove that:

a) if $\dfrac{AM}{MB} = \dfrac{BN}{NC} = \dfrac{CP}{PA}$, then T is the centroid of $\triangle ABC$;

b) $S_{MNP} \leq \dfrac{1}{3}S_{ABC}$.

Sava Grozdev, Svetlozar Doychev

Problem 12.3. In a group of n people there are three that are familiar to each other and any of them is familiar with more then the half of the people in the group. Find the minimum possible triples of familiar people?

Nickolay Khadzhiivanov

Spring Mathematical Competition
Kazanlak, March 28-30, 2003

Problem 8.1. Is it possible to write the integers $1, 2, 3, 4, 5, 6, 7, 8$ at the vertices of a regular octagon such that the sum of the integers in any three consecutive vertices is greater than:

a) 13; b) 11; c) 12?

Ivan Tonov

Problem 8.2. Let A_1, B_1 and C_1 be respectively the midpoints of the sides BC, CA and AB of $\triangle ABC$ with centroid M. The line trough A_1 and parallel to BB_1 meets the line B_1C_1 at a point D. Prove that if the points A, B_1, M and C_1 are concyclic, then $\sphericalangle ADA_1 = \sphericalangle CAB$.

Chavdar Lozanov

Problem 8.3. Find the least positive integer m such that 2^{2000} divides $2003^m - 1$.

Ivan Tonov

Problem 9.1. Find all real values of a such that the system

$$\left| \begin{array}{l} \dfrac{ax+y}{y+1} + \dfrac{ay+x}{x+1} = a \\ ax^2 + ay^2 = (a-2)xy - x \end{array} \right.$$

has a unique solution.

Peter Boyvalenkov

Problem 9.2. Let $ABCD$ be a parallelogram and let $\sphericalangle BAD$ be acute. Denote by E and F the feet of the perpendiculars from the vertex C to the lines AB and AD, respectively. A circle through D and F is tangent to the diagonal AC at a point Q and a circle through B and E is tangent to the segment QC at its midpoint P. Find the length of diagonal AC if $AQ = 1$.

Ivaylo Kortezov

Problem 9.3. The dragon Spas has one head. His family tree consists of Spas, the Spas parents, their parents, etc. It is known that if a dragon has n heads, then his mother has $3n$ heads and his father has $3n+1$ heads. A positive integer is called *good* if it can be written in a unique way as a sum of the numbers of the heads of two dragons from the Spas' family tree. Prove that 2003 is a good number and find the number of the good numbers less than 2003.

Ivaylo Kortezov

Problem 10.1. a) Find the image of the function $\dfrac{x^2}{x-1}$.

b) Find all real numbers a such that the equation

$$x^4 - ax^3 + (a+1)x^2 - 2x + 1 = 0$$

has no real roots.

Aleksander Ivanov

Problem 10.2. Three nonintersecting circles $k_i(O_i, r_i)$, $i = 1, 2, 3$, where $r_1 < r_2 < r_3$, are tangent to the arms of an angle. One of the arms is tangent to k_1 and k_3 at points A and B and the other one is tangent to k_2 at point C. Let $K = AC \cap k_1$, $L = AC \cap k_2$, $M = BC \cap k_2$ and $N = BC \cap k_3$. The four lines through C and $P = AM \cap BK$, $Q = AM \cap BL$, $R = AN \cap BK$ and $S = AN \cap BL$, meet AB at the points X, Y, Z and T, respectively. Prove that $XZ = YT$.

Emil Kolev

Problem 10.3. Three of n equal balls are radioactive. A detector measures radioactivity. Any measurement of a set of balls gives as a result whether 0, 1 or more than 1 balls are radioactive. Denote by $L(n)$ the least number of measurements that one needs to find the three radioactive balls.

a) Find $L(6)$.

b) Prove that $L(n) \leq \dfrac{n+5}{2}$.

Emil Kolev

Problem 11.1. Let $a \geq 2$ be a real number. Denote by x_1 and x_2 the roots of the equation $x^2 - ax + 1 = 0$ and set $S_n = x_1^n + x_2^n$, $n = 1, 2, \ldots$.

a) Prove that the sequence $\left\{\dfrac{S_n}{S_{n+1}}\right\}_{n=1}^{\infty}$ is decreasing.

b) Find all a such that

$$\frac{S_1}{S_2} + \frac{S_2}{S_3} + \cdots + \frac{S_n}{S_{n+1}} > n - 1$$

for any $n = 1, 2, \ldots$.

Oleg Mushkarov

Problem 11.2. The incircle of $\triangle ABC$ has radius r and is tangent to the sides AB, BC and CA at points C_1, A_1 and B_1, respectively. If $N = BC \cap B_1C_1$ and $AA_1 = 2A_1N = 2r\sqrt{3}$, find $\sphericalangle ANC$.

Sava Grozdev, Svetlozar Doychev

Problem 11.3. Find all positive integers n for which there exists n points in the plane such that any of them lies on exactly $\frac{1}{3}$ of the lines determined by these n points.

Aleksander Ivanov, Emil Kolev

Problem 12.1. Consider the functions

$$f(x) = \frac{\cos^2 x}{1 + \cos x + \cos^2 x} \text{ and } g(x) = k \tan x + (1 - k)\sin x - x,$$

where k is a real number.

a) Prove that $g'(x) = \dfrac{(1-\cos x)(k - f(x))}{f(x)}$.

b) Find the image of $f(x)$ if $x \in \left[0; \dfrac{\pi}{2}\right)$.

c) Find all k such that $g(x) \geq 0$ for any $x \in \left[0; \dfrac{\pi}{2}\right)$.

Sava Grozdev, Svetlozar Doychev

Problem 12.2. Let M be the centroid of $\triangle ABC$ with $\measuredangle AMB = 2\measuredangle ACB$. Prove that:

a) $AB^4 = AC^4 + BC^4 - AC^2.BC^2$;

b) $\measuredangle ACB \geq 60^\circ$.

Nikolai Nikolov

Problem 12.3. Let $\mathbb{R}$ be the set of real numbers. Find all $a > 0$ such that there exists a function $f : \mathbb{R} \to \mathbb{R}$ with the following two properties:

a) $f(x) = ax + 1 - a$ for any $x \in [2, 3)$;

b) $f(f(x)) = 3 - 2x$ for any $x \in \mathbb{R}$.

Oleg Mushkarov, Nikolai Nikolov

52. Bulgarian Mathematical Olympiad
Regional round, April 19-20, 2003

Problem 1. A right-angled trapezoid with area 10 and altitude 4 is divided into two circumscribed trapezoids by a line parallel to its bases. Find their inradii.

Oleg Mushkarov

Problem 2. Let n be a positive integer. Ann writes down n different positive integers. Then Ivo deletes some of them (possible none, but not all), puts the signs $+$ or $-$ in front of each of the remaining numbers and sums them up. Ivo wins if 2003 divides the result; otherwise, Ann wins. Who has a winning strategy?

Ivailo Kortezov

Problem 3. Find all real numbers a such that $4[an] = n + [a[an]]$ for any positive integer n ($[x]$ denotes the largest integer less than or equal to x).

Nikolai Nikolov

Problem 4. Let D be a point on the side AC of $\triangle ABC$ such that $BD = CD$. A line parallel to BD intersects the sides BC and AB at points E and F, respectively. Set $G = AE \cap BD$. Prove that $\sphericalangle BCG = \sphericalangle BCF$.

Oleg Mushkarov, Nikolai Nikolov

Problem 5. Find the number of real solution of the system

$$\left|\begin{array}{l} x + y + z = 3xy \\ x^2 + y^2 + z^2 = 3xz \\ x^3 + y^3 + z^3 = 3yz. \end{array}\right.$$

Sava Grozdev, Svetlozar Doychev

Problem 6. A set C of positive integers is called *good* if for any integer k there exist $a, b \in C$, $a \neq b$, such that the numbers $a + k$ and $b + k$ are not coprime. Prove that if the sum of the elements of a good set C equals 2003, then there exists $c \in C$ for which the set $C \setminus \{c\}$ is good.

Alexander Ivanov, Emil Kolev

52. Bulgarian Mathematical Olympiad
National round, Sofia, May 17-18, 2003

Problem 1. Find the least positive integer n with the following property: if n distinct sums of the form $x_p + x_q + x_r$, $1 \le p < q < r \le 5$, equal 0, then $x_1 = x_2 = x_3 = x_4 = x_5 = 0$.

Sava Grozdev, Svetlozar Doychev

Problem 2. Let H be a point on the altitude CP ($P \in AB$) of an acute $\triangle ABC$. The lines AH and BH intersect BC and AC at points M and N, respectively.

a) Prove that $\sphericalangle MPC = \sphericalangle NPC$.

b) The lines MN and CP intersect at O. A line through O meets the sides of the quadrilateral $CNHM$ at points D and E. Prove that $\sphericalangle DPC = \sphericalangle EPC$.

Alexander Ivanov

Problem 3. Consider the sequence

$$y_1 = y_2 = 1, \; y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k, \; n \ge 1.$$

Find all integers k such that any term of the sequence is a perfect square.

Sava Grozdev, Svetlozar Doychev

Problem 4. A set of at least three positive integers is called *uniform* if removing any of its elements the remaining set can be disjoint into two subsets with equal sums of elements. Find the minimal cardinality of a uniform set.

Peter Boyvalenkov, Emil Kolev

Problem 5. Let a, b and c be rational numbers such that $a + b + c$ and $a^2 + b^2 + c^2$ are equal integers. Prove that the number abc can be written as a ratio of a perfect cube and a perfect square that are coprime.

Oleg Mushkarov, Nikolai Nikolov

Problem 6. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer n the equation $P(x) = 2^n$ has an integer solution.

Oleg Mushkarov, Nikolai Nikolov

Team selection test for 20. BMO
Kazanlak, March 3, 2003

Problem 1. Let D be a point on the side AC of $\triangle ABC$ with $AC = BC$, and E be a point on the segment BD. Prove that $\sphericalangle EDC = 2\sphericalangle CED$ if $BD = 2AD = 4BE$.

Mediteranian Mathematical Competition

Problem 2. Prove that if a, b and c are positive numbers with sum 3, then

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \ge \frac{3}{2}.$$

Mediteranian Mathematical Competition

Problem 3. At any lattice point in the plane a number from the interval $(0, 1)$ is written. It is known that for any lattice point the number written there is equal to the arithmetic mean of the numbers written at the four closest lattice points. Prove that all written numbers are equal.

Mediteranian Mathematical Competition

Problem 4. For any positive integer n set

$$A_n = \{j : 1 \le j \le n,\ (j, n) = 1\}.$$

Find all n such that the polynomial

$$P_n(x) = \sum_{j \in A_n} x^{j-1}$$

is irreducible over $\mathbb{Z}[x]$.

Team selection test for 44. IMO
Sofia, May 29-30, 2003

Problem 1. Cut 2003 rectangles from an acute $\triangle ABC$ such that any of them has a side parallel to AB and the sum of their areas is maximal.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y + f(y)) = 2y + (f(x))^2$$

for any $x, y \in \mathbb{R}$.

Problem 3. Some of the vertices of a convex n-gon are connected by segments such that any two of them have no a common interior point. Prove that for any n points in general position (i.e., any three of them are not collinear) there is an one-to-one correspondence between the points and the vertices of the n-gon such that any two segments corresponding to the respective segments from the n-gon have no a common interior point.

Problem 4. Is it true that for any permutation $a_1, a_2, \dots, a_{2002}$ of $1, 2, \dots, 2002$ there are positive integers m and n of the same parity such that $1 \le m < n \le 2002$ and $a_m + a_n = 2a_{\frac{m+n}{2}}$.

Problem 5. Let $ABCD$ be a circumscribed quadrilateral and let P be the orthogonal projection of its incenter on the diagonal AC. Prove that $\sphericalangle APB = \sphericalangle APD$.

Problem 6. Prove that there are no positive integers m and n such that

$$m(m+1)(m+2)(m+3) = n(n+1)^2(n+2)^3(n+3)^4.$$

Bulgarian Mathematical Competitions 2004

Winter Mathematical Competition

Rousse, January 30 - February 1, 2004

Problem 9.1. Find all values of a such that the equation

$$(a^2 - a - 9)x^2 - 6x - a = 0$$

has two distinct positive roots.

Ivan Landjev

Problem 9.2. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ with circumcenter I intersect at a point E. If the midpoints of segments AD, BC and IE are collinear, prove that $AB = CD$.

Stoyan Atanasov

Problem 9.3. Find the least number of colors with the following property: the integers $1, 2 \ldots, 2004$ can be colored such that there are no integers $a < b < c$ of the same color for which a divides b and b divides c.

Alexander Ivanov

Problem 10.1. Let $f(x) = x^4 - x^3 + 8ax^2 - ax + a^2$ and $g(y) = y^2 - y + 6a$.

a) Prove that $f(x) = (x^2 - y_1x + a)(x^2 - y_2x + a)$, where y_1 and y_2 are the roots of the equation $g(y) = 0$.

b) Find all values of a such that the equation $f(x) = 0$ has for distinct positive roots.

Kerope Tchakerian

Problem 10.2. Let $ABCDE$ be a cyclic pentagon with $AC||DE$. Denote by M the midpoint of BD. If $\sphericalangle AMB = \sphericalangle BMC$, prove that BE bisects AC.

Peter Boyvalenkov

Problem 10.3. Find the largest positive integer n for which there exists a set $\{a_1, a_2, \ldots, a_n\}$ of composite positive integers with the following properties:

(i) any two of them are coprime;

(ii) $1 < a_i \leq (3n + 1)^2$ for $i = 1, \ldots, n$.

Ivan Landjev

Problem 11.1. Find all values of a such that the equation

$$4^x - (a^2 + 3a - 2)2^x + 3a^3 - 2a^2 = 0$$

has a unique solution.

Alexander Ivanov, Emil Kolev

Problem 11.2. The point M on the side AB of $\triangle ABC$ is such that the inradii of $\triangle AMC$ and $\triangle BMC$ are equal. The incircles of $\triangle AMC$ and $\triangle BMC$

have centers O_1 and O_2, and are tangent to the side AB at points P and Q, respectively. It is known that $S_{ABC} = 6S_{PQO_2O_1}$.

a) Prove that $10CM + 5AB = 7(AC + BC)$.

b) Find the ratio $\dfrac{AC + BC}{AB}$.

Emil Kolev

Problem 11.3. Let $a > 1$ be a positive integer. The sequence $a_1, a_2, \dots, a_n, \dots$ is defined by $a_1 = 1$, $a_2 = a$ and $a_{n+2} = a.a_{n+1} - a_n$ for $n \geq 1$. Prove that the prime factors of its terms are infinitely many.

Alexander Ivanov

Problem 12.1. Let $a_1 > 0$ and $a_{n+1} = a_n + \dfrac{n}{a_n}$ for $n \geq 1$. Prove that:

a) $a_n \geq n$ for $n \geq 2$;

b) the sequence $\left\{\dfrac{a_n}{n}\right\}_{n\geq 1}$ converges and find its limit.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.2. In triangle ABC with orthocenter H one has that

$$AH.BH.CH = 3 \quad \text{and} \quad AH^2 + BH^2 + CH^2 = 7.$$

Find:

a) the circumradius of $\triangle ABC$;

b) the sides of $\triangle ABC$ with maximum possible area.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.3. Prove that for any integer $a \geq 4$ there exist infinitely many squarefree positive integers n that divide $a^n - 1$.

Oleg Mushkarov, Nikolai Nikolov

Spring Mathematical Competition
Yambol, March 30 - April 1, 2004

Problem 8.1. The bisectors of $\sphericalangle A$, $\sphericalangle B$ and $\sphericalangle C$ of $\triangle ABC$ meet its circumcircle at points A_1, B_1 and C_1, respectively. Set $AA_1 \cap CC_1 = I$, $AA_1 \cap BC = N$ and $BB_1 \cap A_1C_1 = P$. Denote by O the circumcenter of $\triangle IPC_1$ and let $OP \cap BC = M$. If $BM = MN$ and $\sphericalangle BAC = 2 \sphericalangle ABC$, find the angles of $\triangle ABC$.

Chavdar Lozanov

Problem 8.2. In a volleyball tournament for the Euro- African cup the European teams are 9 more than the African teams. Every two teams met exactly once and the European teams gained 9 times more points than the African teams (the winner takes 1 point and the loser takes 0 point). What are the maximum possible points gained by an African team?

Ivan Tonov

Problem 8.3. In every cell of an $n \times n$ table one of the numbers -1, 0 and 1 is written. Is it possible the sums of the numbers in every row and every column to be $2n$ mutually different numbers, if:

a) $n = 4$; b) $n = 5$?

Ivan Tonov

Problem 9.1. Consider the system $\left| \begin{array}{l} x^2 + y^2 = a^2 + 2 \\ \dfrac{1}{x} + \dfrac{1}{y} = a \end{array} \right.$, where a is a real number.

a) Solve the system for $a = 0$.

b) Find all a, for which the system has exactly two solutions.

Svetlozar Doychev, Sava Grozdev

Problem 9.2. Let I be the incenter of $\triangle ABC$ and M be the midpoint of the side AB. Find the least possible value of $\sphericalangle CIM$ if $CI = MI$.

Svetlozar Doychev, Sava Grozdev

Problem 9.3. Find all odd prime numbers p which divide the number $1^{p-1} + 2^{p-1} + \cdots + 2004^{p-1}$.

Kerope Tchakerian

Problem 10.1. Let $f(x) = x^2 - ax + a^2 - 4$, where a is a real number. Find all a, for which:

a) the equation $f(x) = 0$ has two real roots x_1 and x_2 such that $|x_1^3 - x_2^3| \leq 4$;

b) the inequality $f(x) \geq 0$ holds for all integers x.

Peter Boyvalenkov

Problem 10.2. Let $ABCD$ be a cyclic quadrilateral. Denote by I and J the incenters of $\triangle ABD$ and $\triangle BCD$. Prove that $ABCD$ is a circumsribed

quadrilateral if and only if the points A, I, J and C are either collinear or concyclic.

Stoyan Atanasov

Problem 10.3. See Problem 9.3.

Problem 11.1. Find all real numbers a such that the equation

$$\log_{4ax}(x-3a)+\frac{1}{2}\log_{x-3a}4ax=\frac{3}{2}$$

has exactly two solutions.

Emil Kolev

Problem 11.2. Let AA_1, BB_1 and CC_1 be the altitudes of an acute $\triangle ABC$ ($A_1 \in BC$, $B_1 \in CA$ and $C_1 \in AB$). Denote by O the circumcenter of $\triangle ABC$, and by H_1 the orthocenter of $\triangle A_1B_1C_1$. Prove that the midpoint of the segment OH_1 coincides with incenter of the triangle with vertices at the midpoints of the sides of $\triangle A_1B_1C_1$.

Alexander Ivanov

Problem 11.3. Let k be an integer, $1 < k < 100$. For every permutation $a_1, a_2, \dots, a_{100}$ of the integers $1, 2, \dots, 100$, set $a_{101} = 0$ and choose the least integer $m > k$ such that a_m is less than at least $k-1$ of the numbers $a_1, a_2, \dots, a_k$. Find all k for which the number of permutations with $a_m = 1$, is equal to $\frac{100!}{4}$.

Peter Boyvalenkov, Emil Kolev, Nikolai Nikolov

Problem 12.1. Find all real numbers a such that the graphs of the functions x^2-2ax and $-x^2-1$ have two common tangent lines and the perimeter of the quadrilateral with vertices at the tangent points is equal to 6.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.2. The incircle of $\triangle ABC$ is tangent to the sides AC and BC, $AC \neq BC$, at points P and Q, respectively. The excircles to the sides AC и BC are tangent to the line AB at points M and N. Find $\sphericalangle ACB$ if the points M, N, P and Q are concyclic.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.3. See Problem 11.3.

53. Bulgarian Mathematical Olympiad
Regional round, April 17-18, 2004

Problem 9.1. Find all values of a such that the equation

$$\sqrt{(4a^2-4a-1)x^2-2ax+1}=1-ax-x^2$$

has exactly two solutions.

Sava Grozdev, Svetlozar Doychev

Problem 9.2. Let A_1 and B_1 be points on the sides AC and BC of $\triangle ABC$ such that $4AA_1.BB_1 = AB^2$. If $AC = BC$, prove that the line AB and the bisectors of $\sphericalangle AA_1B_1$ and $\sphericalangle BB_1A_1$ are concurrent.

Sava Grozdev, Svetlozar Doychev

Problem 9.3. Let $a, b, c > 0$ and $a + b + c = 1$. Prove that

$$\frac{9}{10} \le \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} < 1.$$

Sava Grozdev, Svetlozar Doychev

Problem 9.4. Solve in integers the equation

$$x^3 + 10x - 1 = y^3 + 6y^2.$$

Sava Grozdev, Svetlozar Doychev

Problem 9.5. A square $n \times n$ ($n \ge 2$) is divided into n^2 unit squares colored in black or white such that the squares at the four corners of any rectangle (containing at least four squares) have no the same color. Find the maximum possible value of n.

Sava Grozdev, Svetlozar Doychev

Problem 9.6. Consider the equations

$$[x]^3 + x^2 = x^3 + [x]^2 \quad \text{and} \quad [x^3] + x^2 = x^3 + [x^2],$$

where $[t]$ is the greatest integer that does not exceed t. Prove that:

a) any solution of the first equation is an integer;

b) the second equation has a non-integer solution.

Sava Grozdev, Svetlozar Doychev

Problem 10.1. Solve the inequality

$$\sqrt{x^2-1} + \sqrt{2x^2-3} + x\sqrt{3} > 0.$$

Peter Boyvalenkov

Problem 10.2. Let M be the centroid of $\triangle ABC$. Prove that:

a) $\cot \sphericalangle AMB = \dfrac{BC^2 + CA^2 - 5AB^2}{12S_{ABC}}$.

b) $\cot \measuredangle AMB + \cot \measuredangle BMC + \cot \measuredangle CMA \le -\sqrt{3}$.

Peter Boyvalenkov

Problem 10.3. In a school there are m boys and j girls, $m \ge 1$, $1 \le j < 2004$. Every student has sent a post card to every student. It is known that the number of the post cards sent by the boys is equal to the number of the post cards sent by girl to girl. Find all possible values of j.

Ivailo Kortezov

Problem 10.4. Consider the function

$$f(x) = (a^2 + 4a + 2)x^3 + (a^3 + 4a^2 + a + 1)x^2 + (2a - a^2)x + a^2,$$

where a is a real parameter.

a) Prove that $f(-a) = 0$.

b) Find all values of a such that the equation $f(x) = 0$ has three different positive roots.

Ivan Landjev

Problem 10.5. Let O and G be respectively the circumcenter and the centroid of $\triangle ABC$ and let M be the midpoint of the side AB. If $OG \perp CM$, prove that $\triangle ABC$ is isosceles.

Ivailo Kortezov

Problem 10.6. Prove that any graph with 10 vertices and 26 edges contains at least 4 triangles.

Ivan Landjev

Problem 11.1. Find all values of $x \in (-\pi, \pi)$ such that the numbers $2^{\sin x}$, $2 - 2^{\sin x + \cos x}$ and $2^{\cos x}$ are consecutive terms of a geometric progression.

Emil Kolev

Problem 11.2. The lines through the vertices A and B that are tangent to the circumcircle of an acute $\triangle ABC$ meet at a point D. If M is the midpoint of the side AB, prove that $\measuredangle ACM = \measuredangle BCD$.

Emil Kolev

Problem 11.3. Let $m \ge 3$ and $n \ge 2$ be integers. Prove that in a group of $N = mn - n + 1$ people such that there are two familiar people among any m, there is a person who is familiar with n people. Does the statement remain true if $N < mn - n + 1$?

Alexander Ivanov

Problem 11.4. The points D and E lie respectively on the perpendicular bisectors of the sides AB and BC of $\triangle ABC$. It is known that D is an interior point for $\triangle ABC$, E does not and $\measuredangle ADB = \measuredangle CEB$. If the line AE meets the segment CD at a point O, prove that the areas of $\triangle ACO$ and the quadrilateral $DBEO$ are equal.

Emil Kolev

Problem 11.5. Let a, b and c be positive integers such that one of them is coprime with any of the other two. Prove that there are positive integers x, y and z such that $x^a = y^b + z^c$.

Alexander Ivanov

Problem 11.6. One chooses a point in the interior of $\triangle ABC$ with area 1 and connects it with the vertices of the triangle. Then one chooses a point in the interior of one of the three new triangles and connects it with its vertices, etc. At any step one chooses a point in the interior of one of the triangles obtained before and connects it with the vertices of this triangle. Prove that after the n-th step:

a) $\triangle ABC$ is divided into $2n+1$ triangles;

б) there are two triangles with common side whose combined area is not less than $\frac{2}{2n+1}$.

Alexander Ivanov

Problem 12.1. Solve in integers the equation

$$2^a + 8b^2 - 3^c = 283.$$

Oleg Mushkarov, Nikolai Nikolov

Problem 12.2. Find all values of a such that the maximum of the function $f(x) = \frac{ax-1}{x^4 - x^2 + 1}$ is equal to 1.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.3. A plane bisects the volume of the tetrahedron $ABCD$ and meets the edges AB and CD respectively at points M and N such that $\frac{AM}{BM} = \frac{CN}{DN} \neq 1$. Prove that the plane passes through the midpoints of the edges AC and BD.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.4. Let $ABCD$ be a circumscribed quadrilateral. Find $\sphericalangle BCD$ if $AC = BC$, $AD = 5$, $E = AC \cap BD$, $BE = 12$ and $DE = 3$.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.5. A set A of positive integers less than 2 000 000 is called *good* if $2000 \in A$ and a divides b for any $a, b \in A$, $a < b$. Find:

a) the maximum possible cardinality of a good set;

b) the number of the good sets of maximal cardinality.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.6. Find all non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients such that $P(x)Q(x+1) = P(x+2004)Q(x)$ for any x.

Oleg Mushkarov, Nikolai Nikolov

53. Bulgarian Mathematical Olympiad
National Round, Sofia, May 15-16, 2004

Problem 1. Let I be the incenter of $\triangle ABC$ and let A_1, B_1 and C_1 be points on the segments AI, BI and CI. The perpendicular bisectors of the segments AA_1, BB_1 and CC_1 intersect at points A_2, B_2 and C_2. Prove that the circumcenters of $\triangle A_2B_2C_2$ and $\triangle ABC$ coincide if and only if I is the orthocenter of $\triangle A_1B_1C_1$.

Oleg Mushkarov, Nikolai Nikolov

Problem 2. For any positive integer n the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is written in the form $\frac{p_n}{q_n}$, where p_n and q_n are coprime numbers.

a) Prove that 3 does not divide p_{67}.

b) Find all n, for which 3 divides p_n.

Nikolai Nikolov

Problem 3. In a group of n tourists, among every three of them there are at least two that are not familiar. For any partition of the group into two groups, there are at least two familiar tourists in some of the groups. Prove that there is a tourist who is familiar with at most $\frac{2n}{5}$ tourists.

Ivan Landjev

Problem 4. In any word with letters a and b the following changes are allowed: $aba \to b$, $b \to aba$, $bba \to a$ and $a \to bba$. Is it possible to obtain the word $b\underbrace{aa\ldots a}_{2003}$ from the word $\underbrace{aa\ldots a}_{2003}b$?

Emil Kolev

Problem 5. Let a, b, c and d be positive integers such that there are exactly 2004 ordered pairs (x, y), $x, y \in (0, 1)$, for which $ax + by$ and $cx + dy$ are integers. If $(a, c) = 6$, find (b, d).

Oleg Mushkarov, Nikolai Nikolov

Problem 6. Let p be a prime number and let $0 \le a_1 < a_2 < \cdots < a_m < p$ and $0 \le b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Denote by k the number of different remainders of the numbers $a_i + b_j$, $1 \le i \le m$, $1 \le j \le n$, modulo p. Prove that:

a) if $m + n > p$, then $k = p$; b) if $m + n \le p$, then $k \ge m + n - 1$.

Vladimir Barzov, Alexander Ivanov

Team selection test for 21. BMO
Sofia, March 30-31, 2004

Problem 1. Is there a set $A \supset \{1, 2, \ldots, 2004\}$ of positive integers such that the product of its elements is equal to the sum of their squares?

Problem 2. Prove that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \geq 0$ and $c_k = \prod_{i=1}^{k} b_i^{\frac{1}{k}}$, $1 \leq k \leq n$, then

$$nc_n + \sum_{k=1}^{n} k(a_k - 1)c_k \leq \sum_{k=1}^{n} a_k^k b_k.$$

Problem 3. Let $A = \{1, 2, \ldots, n\}$, $n \geq 4$. For any function $f : A \to A$ and any $a \in A$ define $f_1(a) = f(a)$, $f_{i+1}(a) = f(f_i(a))$, $i \geq 1$. Find the number of the functions f such that f_{n-2} is a constant function but f_{n-3} is not.

Problem 4. Let $A_1A_2 \ldots A_n$ be a convex polygon and let p_i be the length of its orthogonal projection on the line A_iA_{i+1}, $1 \leq i \leq n$ $(A_{n+1} \equiv A_1)$. Prove that if $\sum_{i=1}^{n} \frac{A_iA_{i+1}}{p_i} = 4$, then the polygon is a rectangle.

Problem 5. Let $p(x)$ and $q(x)$ be polynomials with $m \geq 2$ non-zero coefficients. If $\frac{p(x)}{q(x)}$ is not a constant function, find the least possible number of the non-zero coefficients of the polynomial $f(u, v) = p(u)q(v) - p(v)q(u)$.

Problem 6. Let M be a point on a circle k. A circle k_1 with center M meets k at points C and D. A chord AB of k is tangent to k_1 at point H. Prove that the line CD bisects the segment MH if and only if AB is a diameter of k.

Problem 7. Let $A_1, A_2, \ldots, A_n$ be finite sets such that

$$|A_i \cap A_{i+1}| > \frac{n-2}{n-1}|A_{i+1}|$$

for any $i = 1, 2, \ldots, n$ $(A_{n+1} \equiv A_1)$. Prove that their intersection is a non-empty set.

Problem 8. Let a, b and n be positive integers. Denote by $K(n)$ the number of the representations of 1 as a sum of n numbers of the form $\frac{1}{k}$, where k is a positive integer. Let $L(a, b)$ be the least positive integer m such that the equation $\sum_{i=1}^{m} \frac{1}{x_i} = \frac{a}{b}$ has a solution in positive integers and set $L(b) = \max\{L(a, b), 1 \leq a \leq b\}$. Prove that the number of the positive divisors of b does not exceed $2L(b) + K(L(b) + 2)$.

Team selection test for 45. IMO
Sofia, May 27-31, 2004

Problem 1. Let n be a positive integer. Find all positive integers m, for which there exists a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$, $a_n \neq 0$, such that $(a_0, a_1, \ldots a_n, m) = 1$ and $f(k)$ divides m for any integer k.

Problem 2. Find all primes $p \geq 3$ such that $p - \left[\frac{p}{q}\right] q$ is a square-free integer for any prime $q < p$.

Problem 3. Find the maximum possible value of the inradius of a triangle with vertices in the interior or on the boundary of a unit square.

Problem 4. Find the maximum possible value of the product of different positive integers with sum 2004.

Problem 5. Let H be the orthocenter of $\triangle ABC$. The points $A_1 \neq A$, $B_1 \neq B$ and $C_1 \neq C$ lie respectively on the circumcircles of $\triangle BCH$, $\triangle CAH$ and $\triangle ABH$, and $A_1H = B_1H = C_1H$. Denote by H_1, H_2 and H_3 the orthocenters of $\triangle A_1BC$, $\triangle B_1CA$ and $\triangle C_1AB$, respectively. Prove that $\triangle A_1B_1C_1$ and $\triangle H_1H_2H_3$ have the same orthocenter.

Problem 6. In any cell of an $n \times n$ table a number is written such that all the rows are different. Prove that one can remove a column such that the rows in the new table are still different.

Problem 7. The points P and Q lie respectively on the diagonals AC and BD of a quadrilateral $ABCD$ and $\frac{AP}{AC} + \frac{BQ}{BD} = 1$. The line PQ meets the sides AD and BC at points M and N. Prove that the circumcircles of the triangles AMP, BNQ, DMQ and CNP are concurrent.

Problem 8. The edges of a graph with $2n$ vertices, $n \geq 4$, are colored in blue and red such that there is no a blue triangle and there is no a red complete subgraph with n vertices. Find the least possible number of the blue edges.

Problem 9. Prove that among any $2n + 1$ irrational numbers there are $n + 1$ numbers such that the sum of any $2, 3, \ldots, n+1$ of them is an irrational number.

Problem 10. Find all $k > 0$ such that there is a function $f : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

a) $f(f(x,y),z) = f(x, f(y,z))$;

b) $f(x,y) = f(y,x)$;

c) $f(x,1) = x$;

d) $f(zx, zy) = z^k f(x,y)$,

for any $x, y, z \in [0,1]$.

Problem 11. Prove that if $a, b, c \geq 1$ and $a + b + c = 9$, then

$$\sqrt{ab + bc + ca} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

Problem 12. A table with m rows and n columns is given. At any move one chooses some empty cells such that any two of them lie in different rows and columns, puts a white piece in any of these cells and then puts a black piece in the cells whose lines and columns contain white pieces. The game is over if it is not possible to make a move. Find the maximum possible number of the white pieces that can be put on the table.

Bulgarian Mathematical Competitions 2005

Winter Mathematical Competition

Bourgas, January 28-30, 2005

Problem 9.1. Find all values of the real parameter a for which the equations $x^2-(2a+1)x+a=0$ and $x^2+(a-4)x+a-1=0$ have real roots x_1, x_2 and x_3, x_4, respectively, such that

$$\frac{x_1}{x_3}+\frac{x_4}{x_2}=\frac{x_1x_4(x_1+x_2+x_3+x_4)}{a}.$$

Peter Boyvalenkov

Problem 9.2. A circle k through the vertices A and B of an acute $\triangle ABC$ meets the sides AC and BC at inner points M and N, respectively. The tangent lines to k at the points M and N meet at point O. Prove that O is the circumcenter of $\triangle CMN$ if and only if AB is a diameter of k.

Peter Boyvalenkov

Problem 9.3. Find all four-digit positive integers m less than 2005 for which there exists a positive integer $n<m$, such that mn is a perfect square and $m-n$ has at most three distinct positive divisors.

Peter Boyvalenkov, Ivailo Kortezov

Problem 9.4. Ivo writes consecutively the integers 1,2,...,100 on 100 cards and gives some of them to Yana. It is known that for every card of Ivo and every card of Yana, the card with the sum of the numbers on the two cards is not in Ivo and the card with the product of these numbers is not in Yana. How many cards does Yana have if the card with number 13 is in Ivo?

Ivailo Kortezov

Problem 10.1. Consider the inequality $|x^2-5x+6|\le x+a$, where a is a real parameter.

a) Solve the inequality for $a=0$.

b) Find the values of a for which the inequality has exactly three integer solutions.

Stoyan Atanassov

Problem 10.2. Let k be the incircle of $\triangle ABC$ with $AC\ne BC$, I be the center of k and let D, E and F be the tangent points of k to the sides AB, BC and AC, respectively.

a) If $S=CI\cap EF$, prove that $\triangle CDI\sim\triangle DSI$.

b) Let M be the second intersection point of k and CD. The tangent line to k at M intersects the line AB at point G. Prove that $GS\perp CI$.

Stoyan Atanassov, Ivan Landjev

Problem 10.3. Solve in integers the equation

$$z^2 + 1 = xy(xy + 2y - 2x - 4).$$

Ivan Landjev

Problem 10.4. In every cell of a table $n \times n$, $n \geq 2$, one of the numbers $+1$ and -1 is written. The cell on i-th row and j-th column is denoted by (i, j), $i, j = 0, 1, \ldots, n-1$. The neighbors of the cell (i, j) are the cells $(i, j-1)$, $(i, j+1)$, $(i-1, j)$ and $(i+1, j)$, where the numbers are taken modulo n. At each step one replaces the number in each cell with the product of the numbers in the four neighbors of that cell. For example,

$$\begin{array}{|c|c|c|}\hline +1 & -1 & +1 \\ \hline +1 & -1 & -1 \\ \hline -1 & +1 & -1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline +1 & -1 & -1 \\ \hline -1 & +1 & +1 \\ \hline -1 & +1 & +1 \\ \hline \end{array}.$$

A table is called "good" if after finitely many steps one obtains the table with $+1$ in every cell. Find all values of n such that every table $n \times n$ is "good".

Ivan Landjev

Problem 11.1. The sum of the first n terms of an arithmetic progression with first term m and difference 2 is equal to the sum of the first n terms of a geometric progression with first term n and ratio 2.

a) Prove that $m + n = 2^m$;

b) Find m and n, if the third term of the geometric progression is equal to the 23-rd term of the arithmetic progression.

Emil Kolev

Problem 11.2. Find all values of the real parameter a such that the equation

$$\lg(ax + 1) = \lg(x - 1) + \lg(2 - x)$$

has exactly one solution.

Aleksander Ivanov

Problem 11.3. In an acute $\triangle ABC$ with $CA \neq CB$ and incenter O denote by A_1 and B_1 the tangent points of its excircles to the sides CB and CA, respectively. The line CO meets the circumcircle of $\triangle ABC$ at point P and the line through P which is perpendicular to CP meets the line AB at point Q. Prove that the lines QO and A_1B_1 are parallel.

Aleksander Ivanov

Problem 11.4. In an internet chess tournament 2005 chess players took part and everyone played one game against any other. After the tournament it appeared that for every two players A and B who had drawn their game every other player had lost his game with A or with B. Prove that if there were at

least two draws in the tournament then the players can be ordered in such a way that everyone has won his game with the next one in the sequence.

Emil Kolev

Problem 12.1. The sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are such that $a_{n+1} = 2b_n - a_n$ and $b_{n+1} = 2a_n - b_n$ for every n. Prove that:

a) $a_{n+1} = 2(a_1 + b_1) - 3a_n$;

b) if $a_n > 0$ for every n, then $a_1 = b_1$.

Nikolai Nikolov

Problem 12.2. A circle through the vertex A of $\triangle ABC$, $AB \neq AC$, meets the sides AB and AC at points M and N, respectively, and the side BC at points P and Q, where Q lies between B and P. Find $\sphericalangle BAC$, if $MP||AC$, $NQ||AB$ and $\dfrac{BP}{CQ} = \dfrac{AB}{AC}$.

Oleg Mushkarov, Nikolai Nikolov

Problem 12.3. Find all values of the real parameter a such that the image of the function

$$f(x) = \frac{\sin^2 x - a}{\sin^3 x - (a^2 + 2)\sin x + 2}$$

contains the interval $\left[\dfrac{1}{2}, 2\right]$.

Nikolai Nikolov

Problem 12.4. Find all triangles ABC with integer sidelengths such that the side AC is equal to the bisector of $\sphericalangle BAC$ and the perimeter of $\triangle ABC$ is equal to $10p$, where p is a prime number.

Oleg Mushkarov

Spring Mathematical Competition
Kazanlak, March 25-27, 2005

Problem 8.1. Solve the equation

$$\left|\left|x-\frac{5}{2}\right|-\frac{3}{2}\right| = |x^2-5x+4|.$$

Ivan Tonov

Problem 8.2. Let k be the circumcircle of $\triangle ABC$ with $\sphericalangle ACB > 90^\circ$, and BD be the diameter of k through B. The circle k_1 with center D and radius DC meets k at point E and AB at point G. If F is the intersection point of GE and BD, prove that $\sphericalangle DCG = \sphericalangle EFD$.

Chavdar Lozanov

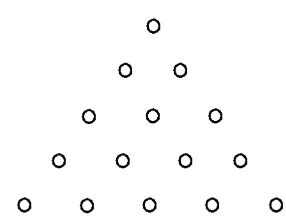
Problem 8.3. Prove that the equation

$$x^2+2y^2+98z^2 = \underbrace{77\ldots7}_{2005}$$

has no integer solutions.

Ivan Tonov

Problem 8.4. Fifteen circles form an equilateral triangle as shown in the figure. Prove that:



a) it is possible to choose 8 circles such that no three of them are vertices of an equilateral triangle;

b) amongst any 9 circles there are three that are vertices of an equilateral triangle.

Ivan Tonov

Problem 9.1. Let $f(x) = x^2+(2a-1)x-a-3$, where a is a real parameter.

a) Prove that the equation $f(x)=0$ has two distinct real roots x_1 and x_2.

b) Find all values of a such that $x_1^3+x_2^3=-72$.

Peter Boyvalenkov

Problem 9.2. A triangle ABC with centroid G and incenter I is given. If $AB=42$, $GI=2$ and $AB||GI$, find AC and BC.

Ivailo Kortezov

Problem 9.3. Four players A_1, A_2, A_3 and A_4 have the same amounts of money and play the following game with seven dices: A_1 throws the seven dices and then pays to each of the other three players $\frac{1}{k}$ of the money that the corresponding player has at the moment, where k is the sum of the points on the seven dices. Then the same action is performed consecutively by A_2, A_3 and A_4 and the game is over. Find the sums of the points on the dices thrown

by each player if after the game their money are in ratio $3:3:2:2$ (the money of A_1 to the money of A_2 to the money of A_3 to the money of A_4).

Peter Boyvalenkov

Problem 9.4. The positive integers M and n are such that M is divisible by all positive integers from 1 to n but it is not divisible by $n+1$, $n+2$ and $n+3$. Find all possible values of n.

Ivailo Kortezov

Problem 10.1. Solve the equation

$$(x+6)5^{1-|x-1|} - x = (x+1)|5^x - 1| + 5^{x+1} + 1.$$

Ivan Landjev

Problem 10.2. Find all values of the real parameter a such that the inequality

$$\sqrt{4+3x} \geq x + a$$

has no an integer solution.

Stoyan Atanassov

Problem 10.3. Let ABC be a triangle with altitude CH, where H is an interior point of the side AB. Denote by P and Q the incenters of $\triangle AHC$ and $\triangle BHC$, respectively. Prove that the quadrilateral $ABQP$ is cyclic if and only if either $AC = BC$ or $\sphericalangle ACB = 90^\circ$.

Stoyan Atanassov

Problem 10.4. Prove that for every positive integer n there exist integers p and q such that

$$|p^2 + 2q^2 - n| \leq \sqrt[4]{9n}.$$

Ivan Landjev

Problem 11.1. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n + 3$, $n \geq 1$.

a) Express a_n as a function of n.

b) Find the limit

$$\lim_{n\to\infty} \frac{\sqrt{a_n} + \sqrt{a_{4n}} + \sqrt{a_{4^2 n}} + \cdots + \sqrt{a_{4^{10} n}}}{\sqrt{a_n} + \sqrt{a_{2n}} + \sqrt{a_{2^2 n}} + \cdots + \sqrt{a_{2^{10} n}}}.$$

Emil Kolev

Problem 11.2. Solve the inequality

$$\log_a(x^2 - x - 2) > \log_a(3 + 2x - x^2)$$

if it is known that $x = a + 1$ is a solution.

Emil Kolev

Problem 11.3. Let M and N be arbitrary points on the side AB of a triangle ABC such that M lies between A and N. The line through M parallel to AC meets the circumcircle of $\triangle MNC$ at point P, and the line through M parallel to NC meets the circumcircle of $\triangle AMC$ at point Q. Analogously, the line through N parallel to BC meets the circumscircle of $\triangle MNC$ at point K and the line through N parallel to MC meets the circumcircle of $\triangle BNC$ at point L. Prove that:

a) the points P, Q and C are collinear;

b) the points P, Q, K and L are concyclic if and only if $AM = BN$.

Alexander Ivanov

Problem 11.4. Let c be a positive integer and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1} < a_n + c$ for every $n \geq 1$. The terms of the sequence are written one after another and in this way one obtains an infinite sequence of digits. Prove that for every positive integer m there exists a positive integer k such that the number formed by the first k digits of the above sequence is divisible by m.

Alexander Ivanov

Problem 12.1. Let ABC be an isosceles triangle such that $AC = BC = 1$ and $AB = 2x$, $x > 0$.

a) Express the inradius r of $\triangle ABC$ as a function of x.

b) Find the maximum possible value of r.

Oleg Mushkarov

Problem 12.2. The excircle to the side AB of a triangle ABC is tangent to the circle with diameter BC. Find $\sphericalangle ACB$ if the lengths of the sides BC, CA and AB form (in this order) an arithmetic progression.

Oleg Mushkarov

Problem 12.3. Find the number of the sequences $\{a_n\}_{n=1}^{\infty}$ of integers such that

$$a_n + a_{n+1} = 2a_{n+2}a_{n+3} + 2005$$

for every n.

Nikolai Nikolov

Problem 12.4. Let $a, b_1, c_1, \ldots, b_n, c_n$ be real numbers such that

$$x^{2n} + ax^{2n-1} + ax^{2n-2} + \cdots + ax + 1 = (x^2 + b_1x + c_1)\ldots(x^2 + b_nx + c_n)$$

for every real number x. Prove that $c_1 = \cdots = c_n = 1$.

Nikolai Nikolov

54. Bulgarian Mathematical Olympiad
Regional round, April 16-17, 2005

Problem 9.1. Find all values of the real parameters a and b such that the remainder in the division of the polynomial $x^4 - 3ax^3 + ax + b$ by the polynomial $x^2 - 1$ is equal to $(a^2+1)x + 3b^2$.

Peter Boyvalenkov

Problem 9.2. Two tangent circles with centers O_1 and O_2 are inscribed in a given angle. Prove that if a third circle with center on the segment O_1O_2 is inscribed in the angle and passes through one of the points O_1 and O_2 then it passes through the other one too.

Peter Boyvalenkov

Problem 9.3. Let a and b be integers and k be a positive integer. Prove that if x and y are consecutive integers such that

$$a^k x - b^k y = a - b,$$

then $|a-b|$ is a perfect k-th power.

Peter Boyvalenkov

Problem 9.4. Find all values of the real parameter p such that the equation $|x^2 - px - 2p + 1| = p - 1$ has four real roots x_1, x_2, x_3 and x_4 such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 20.$$

Ivailo Kortezov

Problem 9.5. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The rays $\overrightarrow{DA}$ and $\overrightarrow{CB}$ meet at point N and the line NT is tangent to k, $T \in k$. The diagonals AC and BD meet at the centroid P of $\triangle NTD$. Find the ratio $NT : AP$.

Ivailo Kortezov

Problem 9.6. A card game is played by five persons. In a group of 25 persons all like to play that game. Find the maximum possible number of games which can be played if no two players are allowed to play simultaneously more than once.

Ivailo Kortezov

Problem 10.1. Solve the system

$$\left|\begin{array}{ccccccc} 3\cdot 4^x & + & 2^{x+1}\cdot 3^y & - & 9^y & = & 0 \\ 2\cdot 4^x & - & 5\cdot 2^x\cdot 3^y & + & 9^y & = & -8 \end{array}\right. .$$

Ivan Landjev

Problem 10.2. Given a quadrilateral $ABCD$ set $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$ and $BD = f$. Prove that:

a) $a^2 + b^2 + c^2 + d^2 \geq e^2 + f^2$;
b) if the quadrilateral $ABCD$ is cyclic then $|a - c| \geq |e - f|$.

Stoyan Atanassov

Problem 10.3. Find all pairs of positive integers (m, n), $m > n$, such that

$$[m^2 + mn, mn - n^2] + [m - n, mn] = 2^{2005},$$

where $[a, b]$ denotes the least common multiple of a and b.

Ivan Landjev

Problem 10.4. Find all values of the real parameter a such that the number of the solutions of the equation

$$3(5x^2 - a^4) - 2x = 2a^2(6x - 1)$$

does not exceed the number of the solutions of the equation

$$2x^3 + 6x = (3^{6a} - 9)\sqrt{2^{8a} - \frac{1}{6}} - (3a - 1)^2 12^x.$$

Ivan Landjev

Problem 10.5. Let H be the orthocenter of $\triangle ABC$, M be the midpoint of AB and H_1 and H_2 be the feet of the perpendiculars from H to the inner and the outer bisector of $\sphericalangle ACB$, respectively. Prove that the points H_1, H_2 and M are colinear.

Stoyan Atanassov

Problem 10.6. Find the largest possible number A having the following property: if the numbers $1, 2, \ldots, 1000$ are ordered in arbitrary way then there exist 50 consecutive numbers with sum not less than A.

Ivan Landjev

Problem 11.1. Find all values of the real parameter a such that the equation

$$a(\sin 2x + 1) + 1 = (a - 3)(\sin x + \cos x)$$

has a solution.

Emil Kolev

Problem 11.2. On the sides of an acute $\triangle ABC$ of area 1 points $A_1 \in BC$, $B_1 \in CA$ and $C_1 \in AB$ are chosen so that

$$\sphericalangle CC_1B = \sphericalangle AA_1C = \sphericalangle BB_1A = \varphi,$$

where the angle φ is acute. The segments AA_1, BB_1 and CC_1 meet at points M, N and P.

a) Prove that the circumcenter of $\triangle MNP$ coincides with the orthocenter of $\triangle ABC$.

b) Find φ, if $S_{MNP} = 2 - \sqrt{3}$.

Emil Kolev

Problem 11.3. Let n be a fixed positive integer. The positive integers a, b, c and d are less than or equal to n, d is the largest one and they satisfy the equality

$$(ab + cd)(bc + ad)(ac + bd) = (d - a)^2(d - b)^2(d - c)^2.$$

a) Prove that $d = a + b + c$.

b) Find the number of the quadruples (a, b, c, d) which have the required properties.

Alexander Ivanov

Problem 11.4. Find all values of the real parameter a such that the equation

$$\log_{ax}(3^x + 4^x) = \log_{(ax)^2}(7^2(4^x - 3^x)) + \log_{(ax)^3} 8^{x-1}$$

has a solution.

Emil Kolev

Problem 11.5. The bisectors of $\sphericalangle BAC$, $\sphericalangle ABC$ and $\sphericalangle ACB$ of $\triangle ABC$ meet its circumcircle at points A_1, B_1 and C_1, respectively. The side AB meets the lines C_1B_1 and C_1A_1 at points M and N, respectively, the side BC meets the lines A_1C_1 and A_1B_1 at points P and Q, respectively, and the side AC meets the lines B_1A_1 and B_1C_1 at points R and S, respectively. Prove that:

a) the altitude of $\triangle CRQ$ through R is equal to the inradius of $\triangle ABC$;

b) the lines MQ, NR and SP are concurrent.

Alexander Ivanov

Problem 11.6. Prove that amongst any 9 vertices of a regular 26-gon there are three which are vertices of an isosceles triangle. Do there exist 8 vertices such that no three of them are vertices of an isosceles triangle?

Alexander Ivanov

Problem 12.1. Prove that if a, b and c are integers such that the number

$$\frac{a(a-b) + b(b-c) + c(c-a)}{2}$$

is a perfect square, then $a = b = c$.

Oleg Mushkarov

Problem 12.2. Find all values of the real parameters a and b such that the graph of the function $y = x^3 + ax + b$ has exactly three common points with the coordinate axes and they are vertices of a right triangle.

Nikolai Nikolov

Problem 12.3. Let $ABCD$ be a convex quadrilateral. The orthogonal projections of D on the lines BC and BA are denoted by A_1 and C_1, respectively.

The segment A_1C_1 meets the diagonal AC at an interior point B_1 such that $DB_1 \geq DA_1$. Prove that the quadrilateral $ABCD$ is cyclic if and only if

$$\frac{BC}{DA_1} + \frac{BA}{DC_1} = \frac{AC}{DB_1}.$$

Nikolai Nikolov

Problem 12.4. The point K on the edge AB of the cube $ABCDA_1B_1C_1D_1$ is such that the angle between the line A_1B and the plane (B_1CK) is equal to $60°$. Find $\tan\alpha$, where α is the angle between the planes (B_1CK) and (ABC).

Oleg Mushkarov

Problem 12.5. Prove that any triangle of area $\sqrt{3}$ can be placed into an infinite band of width $\sqrt{3}$.

Oleg Mushkarov

Problem 12.6. Let m be a positive integer, $A = \{-m, -m+1, \ldots, m-1, m\}$ and $f : A \to A$ be a function such that $f(f(n)) = -n$ for every $n \in A$.

a) Prove that the number m is even.

b) Find the number of all functions $f : A \to A$ with the required property.

Nikolai Nikolov

54. Bulgarian Mathematical Olympiad
National round, Sofia, May 14-15, 2005

Problem 1. Find all triples (x, y, z) of positive integers such that

$$\sqrt{\frac{2005}{x+y}} + \sqrt{\frac{2005}{x+z}} + \sqrt{\frac{2005}{y+z}}$$

is a positive integer.

Oleg Mushkarov

Problem 2. Two circles k_1 and k_2 are externally tangent at point T. A line meets k_1 at points A and B and is tangent to k_2 at point X. The line XT meets k_1 at point S and let C be a point on the arc $\overset{\frown}{TS}$ which does not contain A and B. Let CY be the tangent line to k_2 ($Y \in k_2$) such that the segments CY and ST do not intersect. If I is the intersection point of the lines XY and SC, prove that:

a) the points C, T, Y and I are concyclic;

b) I is the center of the excircle of $\triangle ABC$ tangent to the side BC.

Stoyan Atanassov

Problem 3. Let M be the set of the rational numbers in the interval $(0, 1)$. Does there exist a subset A of M such that every number from M can be represented in a unique way as a sum of one or finitely many distinct numbers from A?

Nikolai Nikolov

Problem 4. Let $\triangle A'B'C$ be the image of $\triangle ABC$ under a rotation with center C. Denote by M, E and F the midpoints of the segments BA', AC and $B'C$, respectively. If $AC \neq BC$ and $EM = FM$, find $\sphericalangle EMF$.

Ivailo Kortezov

Problem 5. Let t, a and b be positive integers. We call a $(t; a, b)$-game the following game with two players: the first player subtracts a or b from t, then the second player subtracts a or b from the number obtained by the first player, then again the first player subtracts a or b from the number obtained by the second player and so on. The player who obtains first a negative number looses the game. Prove that there exist infinitely many t such that the first player has a winning strategy for any $(t; a, b)$-game with $a + b = 2005$.

Emil Kolev

Problem 6. Let a, b and c be positive integers such that ab divides $c(c^2 - c + 1)$ and $a + b$ is divisible by $c^2 + 1$. Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ coincide.

Alexander Ivanov

Team selection test for 22. BMO
Sofia, March 29-30, 2005

Problem 1. Find all positive numbers a and b such that

$$[a[bn]] = n - 1$$

for every positive integer n.

Problem 2. The points P and Q lie in the interior of $\triangle ABC$, $\sphericalangle ACP = \sphericalangle BCQ$ and $\sphericalangle CAP = \sphericalangle BAQ$. The feet of the perpendiculars from P to the lines BC, CA and AB are denoted by D, E and F, respectively. Prove that if $\sphericalangle DEF = 90°$, then Q is the orthocenter of $\triangle BDF$.

Problem 3. Does there exist a strictly increasing sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ such that $a_n \leq n^3$ for every n and every positive integer can be written in a unique way as a difference of two terms of the sequence?

Problem 4. A real number is assigned to every point in the plane. Let $\mathcal{P}$ be a convex n-gon. It is known that for every n-gon similar to $\mathcal{P}$ the sum of the numbers assigned to its vertices is equal to 0. Prove that all numbers assigned to the points in the plane are equal to 0.

Problem 5. If $a_0 = 0$ and $a_n = a_{\left[\frac{n}{2}\right]} + \left[\frac{n}{2}\right]$, $n \geq 1$, find $\lim\limits_{n\to+\infty} \dfrac{a_n}{n}$.

Problem 6. Let $a_1, a_2, \ldots, a_m$ be arbitrary positive integers. Prove that there exist distinct positive integers $b_1, b_2, \ldots, b_n$, $n \leq m$, such that the following two conditions are satisfied:

(1) all subsets of $\{b_1, b_2, \ldots, b_n\}$ have distinct sums of elements;

(2) every number $a_1, a_2, \ldots, a_m$ is the sum of the elements of some subset of $\{b_1, b_2, \ldots, b_n\}$.

Problem 7. The extensions of the sides AB and CD of a convex quadrilateral $ABCD$ meet at point P and the extensions of the sides BC and AD meet at point Q. The point O from the interior of the quadrilateral is such that $\sphericalangle BOP = \sphericalangle DOQ$. Prove that $\sphericalangle AOB + \sphericalangle COD = 180°$.

Problem 8. In a group of B boys and G girls it is known that $G \geq 2B - 1$. Some boys know some girls. Prove that it possible to arrange a dance in pairs in such a way that all boys will dance and every boy who does not know the girl in his pair knows only girls who do not dance.

Team selection test for 46. IMO
Sofia, May 18-19, 2005

Problem 1. Let ABC be an acute triangle. Find the locus of the points M in the interior of $\triangle ABC$ such that

$$AB - FG = \frac{MF.AG + MG.BF}{CM},$$

where F and G are the feet of the perpendiculars from M to the lines BC and AC, respectively.

Peter Boyvalenkov, Nikolai Nikolov

Problem 2. Find the number of the subsets B of the set $\{1, 2, \ldots, 2005\}$ having the following property: the sum of the elements of B is congruent to 2006 modulo 2048.

Emil Kolev

Problem 3. Let $\mathbb{R}^*$ be the set of non-zero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = f^2(x) + \frac{f(xy)}{f(x)}$$

for all $x, y \in \mathbb{R}^*$, $y \neq -x^2$.

Alexander Ivanov

Problem 4. Let $a_1, a_2, \ldots, a_{2005}$, $b_1, b_2, \ldots, b_{2005}$ be real numbers such that the inequality

$$(a_i x - b_i)^2 \geq \sum_{j=1, j \neq i}^{2005} (a_j x - b_j)$$

holds true for every real number x and all $i = 1, 2, \ldots, 2005$. Find the maximum possible number of the positive numbers amongst a_i and b_i, $i = 1, 2, \ldots, 2005$.

Nazar Agakhanov, Nikolai Nikolov

Problem 5. Let ABC ($AC \neq BC$) be an acute triangle with orthocenter H and incenter I. The lines CH and CI meet the circumcircle of $\triangle ABC$ at points D and L, respectively. Prove that $\sphericalangle CIH = 90^\circ$ if and only if $\sphericalangle IDL = 90^\circ$.

Stoyan Atanassov

Problem 6. In a group of 9 persons it is not possible to choose 4 persons such that every one knows the three others. Prove that this group of 9 persons can be partitioned into four parts in such a way that nobody knows anyone from his part.

Emil Kolev

Bulgarian Mathematical Competitions 2006

Winter Mathematical Competition
Pleven, February 3-5, 2006

Problem 9.1. Find all pairs (a, b) of non-negative real numbers such that the equations $x^2 + a^2x + b^3 = 0$ and $x^2 + b^2x + a^3 = 0$ have a common real root.

Peter Boyvalenkov

Problem 9.2. Let b and c be real numbers such that the equation $x^2+bx+c = 0$ has two distinct real roots x_1 and x_2 with $x_1 = x_2^2 + x_2$.

a) Find b and c if $b + c = 4$.

б) Find b and c if they are coprime integers.

Stoyan Atanasov

Problem 9.3. Given a triangle ABC, let BL, $L \in AC$, be the bisector of $\sphericalangle ABC$ and AH, $H \in BC$, the altitude to BC. Prove that $\sphericalangle AHL = \sphericalangle ALB$ if and only if $\sphericalangle BAC = \sphericalangle ACB + 90°$.

Stoyan Atanasov

Problem 9.4. Tokens are placed in some of the cells of a table of size 8×8 such that:

(1) there is at least one token in any rectangle of size 2×1 and 1×2;

(2) there are two neighboring tokens in any rectangle of size 7×1 and 1×7.

Find the minimum possible number of tokens.

Peter Boyvalenkov

Problem 10.1. Consider the inequality $\sqrt{x} + \sqrt{2-x} \geq \sqrt{a}$, where a is a real number.

a) Solve the inequality for $a = 3$.

б) Find all a, for which the set of solutions of the inequality is a segment (possibly, a point) of length less than or equal to $\sqrt{3}$.

Kerope Chakaryan

Problem 10.2. Let $ABCD$ be a parallelogram. The points E and F on the sides AB and BC, respectively, are such that DE is the bisector of $\sphericalangle ADF$ and $AE + CF = DF$. The line through C and perpendicular to DE meets the side AD at L and the diagonal BD at H. Set $N = DE \cap AC$. Prove that:

a) $AE = DL$;

b) $BC = CD$ if $HN||AD$;

c) $ABCD$ is a square if $HN||AD$.

Ivailo Kortezov

Problem 10.3. Find all positive integers t, x, y, z such that

$$2^t = 3^x 5^y + 7^z.$$

Kerope Chakaryan

Problem 10.4. There are 40 knights in a kingdom. Every morning they fight in pairs (everyone has exactly one enemy to fight with) and every evening they sit around a table (during the evening they do not change their sits). Find the least number of days such that:

a) the fights can be arranged in a way that every two knights have fought at least once;

b) round the table arrangements can be done in a way that every two knights have been neighbors around the table.

Ivailo Kortezov

Problem 11.1. Solve the equation

$$\log_a(a^{2(x^2+x)} + a^2) = x^2 + x + \log_a(a^2 + 1),$$

where a is a real number.

Emil Kolev

Problem 11.2. Given a triangle ABC with $\sphericalangle ACB = 60^\circ$, define the sequence of points $A_0, A_1, \ldots, A_{2006}$ in the following way: $A_0 = A$, A_1 is the orthogonal projection of A_0 on BC, A_2 is the orthogonal projection of A_1 on AC, $\ldots$, A_{2005} is the orthogonal projection of A_{2004} on BC and A_{2006} is the orthogonal projection of A_{2005} on AC. The sequence of points $B_0, B_1, \ldots, B_{2006}$ is defined in a similar way: $B_0 = B$, B_1 is the orthogonal projection of B_0 on AC, B_2 is the orthogonal projection of B_1 on BC and so on. Prove that the line $A_{2006}B_{2006}$ is tangent to the incircle of $\triangle ABC$ if and only if

$$\frac{AC + BC}{AB} = \frac{2^{2006} + 1}{2^{2006} - 1}.$$

Aleksandar Ivanov

Problem 11.3. Let a be an integer. Find all real numbers x, y, z such that

$$a(\cos 2x + \cos 2y + \cos 2z) + 2(1 - a)(\cos x + \cos y + \cos z) + 6 = 9a.$$

Aleksandar Ivanov

Problem 11.4. A positive integer a whose decimal representation has 2006 digits is called "bad" if 3 does not divide any integer formed by three consecutive digits of a.

a) Find the number of all bad integers whose digits are equal to 1, 2 or 3.

b) Let a and b be different bad integers such that $a + b$ is also a bad integer. Denote by k the number of positions, where the digits of a and b coincide. Find all possible values of k.

Emil Kolev

Problem 12.1. Consider the function

$$f(x) = \frac{x^2 - 2006x + 1}{x^2 + 1}.$$

a) Solve the inequality $f'(x) \geq 0$.

b) Prove that $|f(x) - f(y)| \leq 2006$ for all real numbers x and y.

Oleg Mushkarov

Problem 12.2. Let k be a circle with center O and radius $\sqrt{5}$ and let M and N be points on a diameter of k such that $MO = NO$. The chords AB and AC, passing through M and N, respectively, are such that

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{3}{MN^2}.$$

Find the length of MO.

Oleg Mushkarov

Problem 12.3. Find the maximal cardinality of a set of phone numbers satisfying the following three conditions:

a) all of them are five-digit numbers (the first digit can be 0);

b) each phone number contains at most two different digits;

c) the deletion of an arbitrary digit in two arbitrary phone numbers (possibly in different positions) does not lead to identical sequences of digits of length 4.

Ivan Landjev

Problem 12.4. Let O be the circumcenter of a triangle ABC with $AC = BC$. The line AO meets the side BC at D. If the lengths of BD and CD are integers, and $AO - CD$ is a prime number, find these three numbers.

Nikolai Nikolov

Spring Mathematical Competition
Yambol, March 24–26, 2006

Problem 8.1. Find all integers a, b, c, d such that $ac-3bd=5$ and $ad+bc=6$.

Ivan Tonov

Problem 8.2. Let A and B be given points on a circle k. For an arbitrary point L on k denote by M the point on the line AL such that $LM = LB$ and L is between A and M. Find the locus of the points M.

Chavdar Lozanov

Problem 8.3. Let m be a positive integer and $u_m = \underbrace{11\dots1}_{m}$. Prove that there is no a positive integer multiple of u_m such that the sum of its digits is less than m.

Ivan Tonov

Problem 8.4. Each side of a sheet of paper is a map of 5 countries. The countries on one of the maps are colored in 5 different colors. Prove that it is possible to color the countries on the other map in such a way that every two are colored in different colors and at least 20% of the sheet is colored in the same color on both sides.

Chavdar Lozanov

Problem 9.1. Find all real numbers a for which the equation $x^2+ax+3a^2-7a-19=0$ has real roots x_1 and x_2 such that

$$\frac{1}{x_1-2}+\frac{1}{x_2-2}=-\frac{2a}{13}.$$

Peter Boyvalenkov

Problem 9.2. In an acute $\triangle ABC$ the altitudes AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) are drawn, I is the incenter and the line CI meets AB at L. It is known that I lies on the circumcircle of $\triangle A_1B_1C$.

a) Prove that L is the center of excircle of $\triangle A_1B_1C$ tangent to the side A_1B_1.

b) If $CI = 2IL$, find $\sphericalangle ACB$.

Stoyan Atanasov

Problem 9.3. The sets $M = \{1, 2, \dots, 27\}$ and $A = \{a_1, a_2, \dots, a_k\} \subset \{1, 2, \dots, 14\}$ have the following property: every element of M is either an element of A or the sum of two (possibly identical) elements of A. Find the minimum value of k.

Peter Boyvalenkov

Problem 9.4. For any positive integer n denote by $f(n)$ the smallest positive integer m such that the sum $1+2+\cdots+m$ is divisible by n. Find all n such that $f(n)=n-1$.

Kerope Chakarian

Problem 10.1. Consider the equations

$$3^{2x+3} - 2^{x+2} = 2^{x+5} - 9^{x+1} \qquad (1)$$

and

$$a.5^{2x} + |a-1|5^x = 1, \qquad (2)$$

where a is a real number.

a) Solve the equation (1).

b) Find the values of a such that the equations (1) and (2) are equivalent.

Kerope Chakarian

Problem 10.2. Let AA', BB' and CC' be the angular bisectors of a triangle ABC with incenter I. The segments CI and $A'B'$ meet at D and the midpoints of the segments AI and BI are denoted by M and N, respectively.

a) If $a = BC$, $b = AC$ and $c = AB$, find the ratio $CD : DI$.

b) If $K = AC \cap \overrightarrow{C'M}$ and $L = BC \cap \overrightarrow{C'N}$, prove that D is the incenter of $\triangle KLC$.

Ivailo Kortezov

Problem 10.3. Forty thieves are to distribute 4000 euro amongst them. A group of five thieves is called *poor* if they have no more than 500 euro all together. What is the minimum number of poor groups amongst all possible groups of five thieves?

Ivailo Kortezov

Problem 10.4. See Problem 9.4.

Problem 11.1. Let $a_1, a_2 \ldots, a_n, \ldots$ be a geometric progression with $a_1 = 3 - 2a$ and ratio $q = \dfrac{3-2a}{a-2}$, where $a \neq \dfrac{3}{2}, 2$ is a real number. Set $S_n = \sum_{i=1}^{n} a_i$, $n \geq 1$. Prove that if the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent and its limit is S, then $S < 1$.

Aleksandar Ivanov

Problem 11.2. Solve the system

$$\left|\begin{array}{l} \left(4^{\sqrt{x^2+x}} + 7.2^{\sqrt{x^2+x}} - 1\right)\sin(\pi y) = 7|\sin(\pi y)| \\ x^2 + 4x + y^2 = 0 \end{array}\right. .$$

Aleksandar Ivanov

Problem 11.3. Consider the excircles of a triangle ABC tangent to the sides AB and AC. Denote by M, N and P the tangent points of the first circle to the side AB and the extensions of the sides BC and CA and by S, Q and R the tangent points of the second circle to the side AC and the extensions of the sides AB and BC. Let X be the intersection point of the lines MN and

RS and Y be the intersection point of the lines PN and RQ. Prove that the points X, A and Y are colinear.

Emil Kolev

Problem 11.4. Let n be a positive integer. Find the number of all finite strictly increasing sequences $a_0 = 1, a_1, \dots, a_k = 2.3^n$ of positive integers with the following property: $\prod_{i=1}^{k}\left[\frac{a_i + a_{i-1} - 1}{a_{i-1}}\right] = 2.3^n$, where $[x]$ is the integral part of x.

Aleksandar Ivanov

Problem 12.1. The sequence $\{x_n\}_{n=1}^{\infty}$ is defined by $x_1 = 2$ and $x_{n+1} = 1+ax_n$, $n \geq 1$, where a is a real number. Find all values of a for which the sequence is:

a) an arithmetic progression;

b) convergent and find its limit.

Oleg Mushkarov

Problem 12.2. Let ABC be a right triangle and D be a point on the hypothenuse AB.

a) Prove that the expression $\frac{AC^2}{AD+CD} + \frac{BC^2}{BD+CD}$ does not depend on D.

b) Let DE ($E \in AC$) and DF ($F \in BC$) be the bisectors of $\sphericalangle ADC$ and $\sphericalangle BDC$, respectively. Find the minimum value of the expression $\frac{CF}{CA} + \frac{CE}{CB}$.

Oleg Mushkarov

Problem 12.3. Find all complex numbers $a \neq 0$ and b such that for every complex root w of the equation $z^4 - az^3 - bz - 1 = 0$ the inequality $|a-w| \geq |w|$ holds.

Nikolai Nikolov

Problem 12.4. See Problem 11.4.

55. Bulgarian Mathematical Olympiad
Regional round, April 15-16, 2006

Problem 9.1. Find all real numbers a such that the roots x_1 and x_2 of the equation

$$x^2 + 6x + 6a - a^2 = 0$$

satisfy the relation $x_2 = x_1^3 - 8x_1$.

Ivan Landjev

Problem 9.2. Two circles k_1 and k_2 meet at points A and B. A line through B meets the circles k_1 and k_2 at points X and Y, respectively. The tangent lines to k_1 at X and to k_2 at Y meet at C. Prove that:

a) $\sphericalangle XAC = \sphericalangle BAY$.

b) $\sphericalangle XBA = \sphericalangle XBC$, if B is the midpoint of XY.

Stoyan Atanasov

Problem 9.3. The positive integers l, m, n are such that $m - n$ is a prime number and $8(l^2 - mn) = 2(m^2 + n^2) + 5(m + n)l$. Prove that $11l + 3$ is a perfect square.

Ivan Landjev

Problem 9.4. Find all integers a such that the equation

$$x^4 + 2x^3 + (a^2 - a - 9)x^2 - 4x + 4 = 0$$

has at least one real root.

Stoyan Atanasov

Problem 9.5. Given a right triangle ABC ($\sphericalangle ACB = 90°$), let CH, $H \in AB$, be the altitude to AB and P and Q be the tangent points of the incircle of $\triangle ABC$ to AC and BC, respectively. If $AQ \perp HP$ find the ratio $\frac{AH}{BH}$.

Stoyan Atanasov

Problem 9.6. An air company operates 36 airlines in a country with 16 airports. Prove that one can make a round trip that includes 4 airports.

Ivan Landjev

Problem 10.1. A circle k is tangent to the arms of an acute angle AOB at points A and B. Let AD be the diameter of k through A and $BP \perp AD$, $P \in AD$. The line OD meets BP at point M. Find the ratio $\frac{BM}{BP}$.

Peter Boyvalenkov

Problem 10.2. Find the maximum of the function

$$f(x) = \frac{\lg x . \lg x^2 + \lg x^3 + 3}{\lg^2 x + \lg x^2 + 2}$$

and the values of x, when it is attained.

Ivailo Kortezov

Problem 10.3. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Find all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{R}$ such that $f(1) = 1$, $f(1/x) = f(x)$ for any $x \in \mathbb{Q}^+$ and $xf(x) = (x+1)f(x-1)$ for any $x \in \mathbb{Q}^+$, $x > 1$.

Ivailo Kortezov

Problem 10.4. The price of a merchandize dropped from March to April by $x\%$, and went up from April to May by $y\%$. It turned out that in the period from March to May the prize dropped by $(y-x)\%$. Find x and y if they are positive integers (the prize is positive for the whole period).

Ivailo Kortezov

Problem 10.5. Let $ABCD$ be a parallelogram such that $\sphericalangle BAD < 90^\circ$ and let DE, $E \in AB$, and DF, $F \in BC$, be the altitudes of the parallelogram. Prove that

$$4(AB.BC.EF + BD.AE.FC) \leq 5.AB.BC.BD.$$

Find $\sphericalangle BAD$ if the equality occurs.

Ivailo Kortezov

Problem 10.6. See problem 9.6.

Problem 11.1. Let k be a circle with diameter AB and let $C \in k$ be an arbitrary point. The excircles of $\triangle ABC$ tangent to the sides AC and BC are tangent to the line AB at points M and N, respectively. Denote by O_1 and O_2 the circumcenters of $\triangle AMC$ and $\triangle BNC$. Prove that the area of $\triangle O_1CO_2$ does not depend on C.

Alexander Ivanov

Problem 11.2. Prove that $t^2(xy + yz + zx) + 2t(x + y + z) + 3 \geq 0$ for all $x, y, z, t \in [-1, 1]$.

Nikolai Nikolov

Problem 11.3. Consider a set S of 2006 points in the plane. A pair $(A, B) \in S \times S$ is called "isolated" if the disk with diameter AB does not contain other points from S. Find the maximum number of "isolated" pairs.

Alexander Ivanov

Problem 11.4. Find the least positive integer a such that the system

$$\left| \begin{array}{l} x + y + z = a \\ x^3 + y^3 + z^2 = a \end{array} \right.$$

has no an integer solution.

Oleg Mushkarov

Problem 11.5. The tangent lines to the circumcircle k of an isosceles $\triangle ABC$, $AC = BC$, at the points B and C meet at point X. If AX meets k at point Y, find the ratio $\frac{AY}{BY}$.

Emil Kolev

Problem 11.6. Let $a_1, a_2, \ldots$ be a sequence of real numbers less than 1 and such that $a_{n+1}(a_n + 2) = 3$, $n \geq 1$. Prove that:

a) $-\frac{7}{2} < a_n < -2$; б) $a_n = -3$ for any n.

Nikolai Nikolov

Problem 12.1. Find the area of the triangle determined by the straight line with equation $x - y + 1 = 0$ and the tangent lines to the graph of the parabola $y = x^2 - 4x + 5$ at its common points with the line.

Emil Kolev

Problem 12.2. See problem 11.5.

Problem 12.3. Find all real numbers a, such that the inequality

$$x^4 + 2ax^3 + a^2x^2 - 4x + 3 > 0$$

holds true for all real numbers x.

Nikolai Nikolov

Problem 12.4. Find all positive integers n for which the equality

$$\frac{\sin(n\alpha)}{\sin\alpha} - \frac{\cos(n\alpha)}{\cos\alpha} = n - 1$$

holds true for all $\alpha \neq \frac{k\pi}{2}$, $k \in Z$.

Emil Kolev

Problem 12.5. A plane intersects a tetrahedron $ABCD$ and divides the medians of the triangles DAB, DBC and DCA through D in ratios $1:2$, $1:3$ and $1:4$ from D, respectively. Find the ratio of the volumes of the two parts of the tetrahedron cut by the plane.

Oleg Mushkarov

Problem 12.6. See problem 11.6.

55. Bulgarian Mathematical Olympiad
National round, Sofia, May 20-21, 2006

Problem 1. Consider the set $A = \{1, 2, 3, 4, \ldots, 2^n\}$, $n \geq 2$. Find the number of the subsets B of A, such that if the sum of two elements of A is a power of 2 then exactly one of them belongs to B.

Aleksandar Ivanov

Problem 2. Let $\mathbb{R}^+$ be the set of all positive real numbers and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that

$$f(x+y) - f(x-y) = 4\sqrt{f(x)f(y)}$$

for all $x > y > 0$.

a) Prove that $f(2x) = 4f(x)$ for all $x \in \mathbb{R}^+$.

b) Find all such functions.

Oleg Mushkarov, Nikolai Nikolov

Problem 3. An infinite sequence of digits is obtained by writing all positive integers one after another in increasing order. Find the least positive integer k such that among the first k digits of the above sequence every two nonzero digits appear different number of times.

Aleksandar Ivanov, Emil Kolev

Problem 4. Let p be a prime number such that p^2 divides $2^{p-1} - 1$. Prove that for any positive integer n the integer $(p-1)(p! + 2^n)$ has at least three distinct prime divisors.

Aleksandar Ivanov

Problem 5. Let ABC be a triangle with $\sphericalangle BAC = 30°$ and $\sphericalangle ABC = 45°$. Consider all pairs of points X and Y such that X and Y lie on the rays $AC^{\rightarrow}$ and $BC^{\rightarrow}$, respectively and $OX = BY$, where O is the circumcenter of $\triangle ABC$. Prove that the perpendicular bisectors of the segments XY pass through a fixed point.

Emil Kolev

Problem 6. Let O be a fixed point in the plane. Find all sets of points S in the plane, containing at least two distinct points, and such that for any point $A \in S$, $A \neq O$, the circle with diameter OA is contained in S.

Nikolai Nikolov, Slavomir Dinev

Team selection test for 23. BMO
Sofia, April 1-2, 2006

Problem 1. Are there exist two triangles whose angles (in some order) form an arithmetic progression with nonzero common difference.

Problem 2. Let CL and CK be the inner and the outer bisectors of angle ACB in $\triangle ABC$, $AC > BC$ and let CM be its median. A point P on CM is such that the points C, A_1, B_1 and P are concyclic, where $A_1 = AP^{\rightarrow} \cap BC$ and $B_1 = BP^{\rightarrow} \cap AC$. Prove that the points C, K, L and P are also concyclic.

Problem 3. Prove that if x, y and a are real numbers from the interval $(0, 1)$, then

$$\frac{|x-y|}{1-xy} \le \frac{|x^a - y^a|}{1-x^a y^a}.$$

Problem 4. Ivan and Peter play the following game. Ivan chooses a secret number from the set $A = \{1, 2, \dots, 90\}$. Then Peter chooses a subset B of A and Ivan tells Peter whether his number is in the set B or not. If the answer is "yes" then Peter pays Ivan 2 leva, and if the answer is "no" then he pays Ivan 1 lev. Find the least amount of leva that Peter needs so that he can always find Ivan's number.

Problem 5. Two real numbers a and b satisfy the inequality $b^3 + b \le a - a^3$. Find the maximum possible value of $a + b$.

Problem 6. Find the number of pairs (m, n) of positive integers such that $m \le 2006$, $n \le 2006$ and the equation

$$(x-m)^{13} = (x-y)^{25} + (y-n)^{37}$$

has an integer solution.

Problem 7. The incircle k of $\triangle ABC$ is tangent to the sides AB, BC and CA at points C_1, A_1 and B_1, respectively. The points C_2, A_2 and B_2 are diametrically opposite to C_1, A_1 and B_1 in k.

a) Prove that the lines AA_2, BB_2 and CC_2 are concurrent.

b) If the line AA_2 meets k at A_3, find the ratio in which the tangent line to k at A_3 divides BC.

Problem 8. After a volleyball tournament (every two teams played exactly once) with n teams it turned out that for any two teams A and B, such that B wins over A, there exist positive integer t and teams $C_1, C_2, \dots, C_t$, such that A wins over C_1, C_1 wins over C_2, ..., C_t wins over B.

Prove that for any $k = 3, 4, \dots, n$ there exist k teams $A_1, A_2, \dots, A_k$, such that A_1 wins over A_2, A_2 wins over A_3, ..., A_{k-1} wins over A_k and A_k wins over A_1.

Team selection test for 47. IMO
Sofia, May 25-30, 2006

Problem 1. In the cells of a square table the numbers 1, 0 or -1 are written in such a way that there is exactly one 1 and exactly one -1 in every row and in every column. Is it always possible to obtain the opposite table by rearranging the rows and the columns of the initial table? (Two tables are called opposite if all the sums of the numbers in the corresponding cells equal 0.)

Emil Kolev

Problem 2. Find all pairs (P, Q) of polynomials with real coefficients such that

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)}$$

for infinitely many $x \in \mathbb{R}$.

Nikolai Nikolov, Oleg Mushkarov

Problem 3. Let ABC be a non-equilateral triangle and let M and N be interior points of it such that $\sphericalangle BAM = \sphericalangle CAN$, $\sphericalangle ABM = \sphericalangle CBN$ and

$$AM.AN.BC = BM.BN.CA = CM.CN.AB = k.$$

Prove that:

a) $3k = AB.BC.CA$;

b) the midpoint of the segment MN is the centroid of $\triangle ABC$.

Nikolai Nikolov

Problem 4. Let k be the circumcircle of $\triangle ABC$ and D be a point on the arc $\overset{\frown}{AB}$, which does not contain C. Denote by I_A and I_B the incenters of $\triangle ADC$ and $\triangle BDC$, respectively. Prove that the circumcircle of $\triangle I_A I_B C$ is tangent to k if and only if

$$\frac{AD}{BD} = \frac{AC + CD}{BC + CD}.$$

Stoyan Atanasov

Problem 5. Prove that if $a, b, c > 0$, then

$$\frac{ab}{3a+4b+5c} + \frac{bc}{3b+4c+5a} + \frac{ca}{3c+4a+5b} \leq \frac{a+b+c}{12}.$$

Nikolai Nikolov

Problem 6. Let $p > 2$ be a prime number. Find the number of the subsets B of the set $\{1, 2, \ldots, p-1\}$ such that p divides the sum of the elements of B.

Ivan Landjev

Problem 7. Let D and E be points on the sides AB and AC of $\triangle ABC$ such that $DE||BC$. The circumcircle k of $\triangle ADE$ meets the segments BE and CD

at points M and N. The lines AM and AN meet BC at points P and Q such that $BC = 2PQ$ and P lies between B and Q. Prove that the circle k, the line BC and the bisector of $\sphericalangle BAC$ are concurrent.

Nikolai Nikolov

Problem 8. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of integers greater than 1 and let $x > 0$ be an irrational number. Denote by x_n the fractional part of the product $a_n a_{n-1} \dots a_1 x$.

a) Prove that $x_n > \dfrac{1}{a_{n+1}}$ for infinitely many n.

b) Find all sequences $\{a_n\}_{n=1}^{\infty}$ such that there exist infinitely many $x \in (0,1)$ for which $x_n > \dfrac{1}{a_{n+1}}$ for all n.

Nikolai Nikolov, Emil Kolev

Problem 9. Let $n \geq 3$ be a positive integer and M be the set of the first n prime numbers. For every nonempty subset X of M denote by $P(X)$ the product of the elements of X. Let N be a set of fractions of the form $\dfrac{P(A)}{P(B)}$, where $A \subset M$, $B \subset M$, $A \cap B = \phi$ such that the product of any 7 elements of N is an integer. What is the maximum possible cardinality of N?

Alexander Ivanov

Problem 10. Find all sequences of positive integers $\{a_n\}_{n=1}^{\infty}$, such that $a_4 = 4$ and the identity

$$\frac{1}{a_1a_2a_3} + \frac{1}{a_2a_3a_4} + \dots + \frac{1}{a_na_{n+1}a_{n+2}} = \frac{(n+3)a_n}{4a_{n+1}a_{n+2}}$$

holds true for every positive integer $n \geq 2$.

Peter Boyvalenkov

Problem 11. Denote by $d(a,b)$ the number of the divisors of a positive integer a, which are greater than or equal to b. Find all positive integers n such that

$$d(3n+1,1) + d(3n+2,2) + \dots + d(4n,n) = 2006.$$

Ivan Landjev

Problem 12. Let $m \geq 5$ and n be positive integers and M be a regular $(2n+1)$-gon. Find the number of convex m-gons with vertices among the vertices of M and having at least one acute angle.

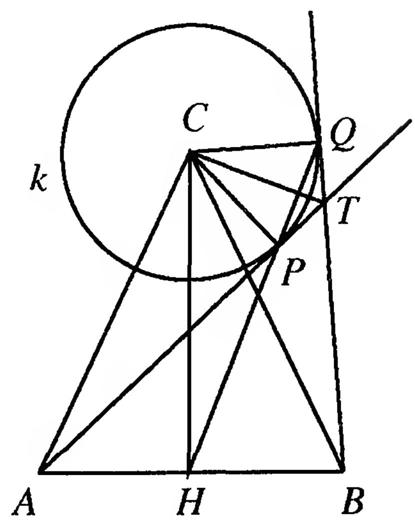
Alexander Ivanov

SOLUTIONS

Bulgarian Mathematical Competitions 2003

Winter Mathematical Competition

9.1. *First solution.* Since $CP = CQ$, $CA = CB$ and $\sphericalangle APC = \sphericalangle BQC = 90°$, then $\triangle APC \cong BQC$. Hence $\sphericalangle CAP = \sphericalangle CBQ$. Setting $AP \cap BQ = T$, it follows that the quadrilateral $ABTC$ is cyclic. Then $\sphericalangle BAC = \sphericalangle QTC$ and now $\sphericalangle TQC = \sphericalangle AHC = 90°$ implies that $\sphericalangle QCT = \sphericalangle ACH$. The equalities $\sphericalangle AHC = \sphericalangle APC = \sphericalangle CPT = \sphericalangle CQT = 90°$ show that $AHPC$ and $CPTQ$ are cyclic quadrilaterals. Thus $\sphericalangle APH = \sphericalangle ACH$ and $\sphericalangle QPT = \sphericalangle QCT$ which means that $\sphericalangle APH = \sphericalangle QPT$. Hence the points H, P and Q are collinear.



Second solution. Set $S = HQ \cap k$. Since the quadrilateral $BHCQ$ is cyclic, and the triangles ABC and CQS are isosceles, it follows that $\sphericalangle BAC = \sphericalangle ABC = \sphericalangle HQC = \sphericalangle CSQ$. Then $AHSC$ is a cyclic quadrilateral and therefore $\sphericalangle ASC = \sphericalangle AHC = 90°$. Hence $S = P$, i.e., the points H, P and Q are collinear.

9.2. The given equation is equivalent to

$$ax^2 + (1-2a)x + (1-a) = 0,$$

where $x \neq -1, -\frac{1}{2}, 1$. Hence this equation has two real roots x_1 and x_2 such that

$$x_2^2 - ax_1 = a^2 - a - 1.$$

Since $x_1 + x_2 = \dfrac{2a-1}{a}$ we get that

$$x_2^2 + ax_2 - a^2 - a + 2 = 0.$$

This together with the identity

$$ax_2^2 + (1-2a)x_2 + 1 - a = 0$$

implies that

$$(a^2 + 2a - 1)x_2 = a^3 + a^2 - 3a + 1 = (a^2 + 2a - 1)(a - 1).$$

The coefficient of x_2 vanishes if $a = -1 \pm \sqrt{2}$.

If $a = -1 + \sqrt{2}$, then

$$(-1 + \sqrt{2})x_2^2 + (3 - 2\sqrt{2})x_2 + (2 - \sqrt{2}) = 0$$

which is impossible, since the discriminant of this quadratic equation equals $33 - 24\sqrt{2} < 0$, i.e. it has no real roots.

If $a = -1 - \sqrt{2}$ we get the equation

$$(-1 - \sqrt{2})x_2^2 + (3 + 2\sqrt{2})x_2 + (2 + \sqrt{2}) = 0$$

that has two real roots, which are not equal to ± 1 and $-\frac{1}{2}$.

Let now $a \neq -1 \pm \sqrt{2}$. Then $x_2 = a - 1$ and hence $a(a-1)(a-3) = 0$. Since $a \neq 0, 1$ we get $a = 3$. In this case the roots of the given equation are $-\frac{1}{3}$ and 2, and they satisfy the given condition.

Thus the desired values of a are $-1 - \sqrt{2}$ and 3.

9.3. We shall prove that the desired number have one of the forms $9k \pm 1$, $3^3(9k \pm 1)$ or $3^6(9k \pm 1)$.

Suppose that 3 does not divide a. Since $n^3 \equiv 0, \pm 1 \pmod 9$, then $a \equiv \pm 1 \pmod 9$.

Conversely, let $a \equiv \pm 1 \pmod 9$. Since 9 divides $1^3 - 1$ and $2^3 + 1$, then there is n_0 such $n_0^3 + a = 3^s t$, where $s \geq 2$ and t is not divisible by 3. We shall prove that if $n_1 = n_0 + 2.3^{s-1}t$, then 3^{s+1} divides $n_1^3 + a$. We have that

$$(n_0 + 2.3^{s-1}t)^3 + a \quad = \quad 3^s t(2n_0^2 + 1) + 4n_0 3^{2s-1}t^2 + 8.3^{3s-3}t^3.$$

Since 3 does not divide n_0, then $2n_0^2+1$ is divisible by 3. Moreover, $2s-1 \geq s+1$ and $3s - 3 \geq s + 1$. Hence $n_1^3 + a$ is divisible by 3^{s+1} but 3 does not divide n_1. Repeating the same argument, we get a positive integer n_p such that 3^{2003} divides $n_p^3 + a$.

Let now 3 divides $a < 2003$. Then $a = 3^s b$, where $s \leq 6$. Hence n is divisible by 3, i.e., $n = 3^p n_0$, where $p \geq 1$ and 3 does not divide n_0. If $p \geq 3$, then 3^9 divides n^3 and does not divide a which implies that 3^{2003} does not divide n^3+a. Hence $p = 1$ or $p = 2$ and it is easy to see that $s = 3$ or $s = 6$, respectively.

In the first case we get that 3^{2000} divides $n_0^3 + b$, where 3 does not divide b and $27b < 2003$. It follows as above that $b \equiv \pm 1 \pmod 9$.

In the second case we get similarly that 3^{1997} divides $n_0^3 + b$, where $729b < 2003$ and $b \equiv \pm 1 \pmod 9$.

The number of the positive integers $b \equiv \pm 1 \pmod 9$ such that $b < 2003$, $27b < 2003$ or $729b < 2003$ equals $2.222 + 1 = 445$, $2.8 + 1 = 17$ or 1, respectively. Hence the desired number is equal to $445 + 17 + 1 = 463$.

10.1. If $ax + 2 < 0$, then the equation has no real roots. If $ax + 2 \geq 0$, it is equivalent to $ax^2 + ax + 2 = (ax + 2)^2$, i.e., $(a^2 - a)x^2 + 3ax + 2 = 0$. The last equation has a unique real root in the following three cases.

Case 1. The coefficient of x^2 vanishes and the respective linear equation has a root x such that $ax + 2 \geq 0$.

If $a = 0$, then $2 = 0$ which is impossible. If $a = 1$, then $x = -\frac{2}{3}$ and $ax + 2 = -\frac{2}{3} + 2 = \frac{4}{3} > 0$. Hence $a = 1$ is a solution of the problem.

Case 2. The coefficient of x^2 is non-zero, i.e., $a \neq 0, 1$, and the respective quadratic equation has a unique real root x with $ax + 2 \geq 0$. Then $D = 9a^2 - 8(a^2 - a) = a^2 + 8a = 0$ and hence $a = -8$. Then $x = \frac{1}{6}$ and $ax + 2 = -8.\frac{1}{6} + 2 = \frac{2}{3} > 0$, i.e., $a = -8$ is a solution of the problem.

Case 3. The coefficient of x^2 is non-zero, i.e., $a \neq 0, 1$, and the respective quadratic equation has two real roots $x_1 < x_2$ such that $ax_1 + 2 < 0 \leq ax_2 + 2$, i.e., $-\frac{2}{a} \in (x_1, x_2]$.

If $-\frac{2}{a} = x_2$, then $(a^2 - a)\left(-\frac{2}{a}\right)^2 + 3a\left(-\frac{2}{a}\right) + 2 = 0$. Hence $-\frac{1}{a} = 0$, a contradiction. Therefore $-\frac{2}{a} \in (x_1, x_2)$ which is equivalent to

$$(a^2 - a)\left((a^2 - a)\left(-\frac{2}{a}\right)^2 + 3a\left(-\frac{2}{a}\right) + 2\right) < 0.$$

It is easy to see that the solutions of the above inequality are $a > 1$. So, the given equation has a unique real root for $a = -8$ and $a \geq 1$.

10.2. a) We shall prove that the position of the point $S = O_1O_2 \cap AB$ does not depend on k. Let O_3 be the center of k. It follows by the Menelaus theorem for $\triangle O_1O_2O_3$ and the line AB that

$$\frac{O_3B}{BO_2} \cdot \frac{O_2S}{SO_1} \cdot \frac{O_1A}{AO_3} = 1.$$

Since $O_3B = AO_3$, we get $\frac{O_2S}{SO_1} = \frac{BO_2}{O_1A} = \frac{R_2}{R_1} = \frac{16}{4} = 4.$

Hence S is a fixed point and the equalities $O_1O_2 = 25 = O_2S + O_1S$ imply that $O_2S = 20$ and $O_1S = 5$.

b) Setting $\sphericalangle O_1O_3O_2 = x$ and $\sphericalangle O_1O_2O_3 = y$, then $\sphericalangle AO_1S = x + y$. Since $\triangle O_1AP$ is isosceles, we have $\sphericalangle APS = \frac{x+y}{2}$. On the other hand, the triangles AO_3B and BO_2Q are also isosceles; hence $\sphericalangle SBO_3 = 90 - \frac{x}{2}$ and $\sphericalangle QBO_2 = 90 - \frac{y}{2}$, which implies that $\sphericalangle SBQ = \frac{x+y}{2}$. Therefore $\sphericalangle APS = \sphericalangle SBQ$, i.e. $PBQA$ is a cyclic quadrilateral.

c) Note that $SP.SQ = SA.SB$ and $SP.SQ = (SO_1 + R_1)(SO_2 - R_2) = 9.4 = 36$. The inequality

$$AB = SA + SB \geq 2\sqrt{SA.SB} = 2\sqrt{SP.SQ} = 12$$

implies that the minimum of AB equals 12 and it is attained if $SA = SB$.

It remains to show that there is a circle k with $SA = SB$. Take a point $A \in k_1$ such that $SA = 6$. Since the power of S with respect to k_1 equals $SO_1^2 - R_1^2 = 5^2 - 4^2 = 9$ and $SA^2 = 36 > 9$, it is easy to see that there is a circle k passing through A and satisfying the conditions of the problem. Then $SB = \sqrt{SP.SQ} = 6$, i.e., $SA = SB$.

10.3. Consider a table with rows corresponding to the elements of A and column corresponding to the elements of M. We write $\times$ in a cell if the element of A in the respective row is adjacent to the element of M in the respective column. Let $|M| = k$. It follows from the given condition that there are no two equal rows which means that M has at least 16 different subsets. Since a set with n elements has 2^n subset, we get $k \geq 4$.

Any row has exactly five adjacent and thus any column contains exactly five $\times$, i.e., the total number of $\times$ is $5k$. A minimal number of $\times$ by rows is attained if one row contains no $\times$, k rows contain one $\times$, $\binom{k}{2}$ rows contain two $\times$, etc.

If $k = 4$, then all the subsets of M are 16 and hence any subset of M appears exactly once as a adjacent set to some element of A. Then we have one row with no $\times$, 4 rows with one $\times$, 6 rows with two $\times$, 4 rows with three $\times$ and one row with four $\times$. The total number of $\times$ becomes $32 > 20$, a contradiction.

For $k = 5$ one has 25 $\times$. Their minimal number by rows is attained when one row contains no $\times$, 5 rows contain one $\times$ and 10 rows – two $\times$. Since $5.1 + 10.2 = 25$ the distribution of $\times$ must be exactly the one described above. This means that any two elements of M are simultaneously adjacent to some element of A.

It is easy to see that if two elements of M do not coincide at at most two positions, then there is no an element of A that is adjacent to them. Hence any two elements of M do not coincide at one or two positions. If there are two elements which do not coincide at one position, we may assume that they are $a = 0000$ и $b = 1000$. The adjacent to a and b in M are 0000 and 1000. So the rows of a and b coincide, a contradiction. This shows that any two elements of M are different at exactly two positions. We may assume that $0000 \in M$. Then the remaining 4 elements are among 0011, 1100, 0101, 1010, 1001 and 0110. But among any of the pairs $(0011, 1100)$, $(0101, 1010)$ and $(1001, 0110)$ at most one element can be chosen, a contradiction.

For $k = 6$ the set $M = \{0000, 1111, 0111, 0100, 1001, 0101\}$ separates any two elements of A.

Hence the desired minimal number is 6.

11.1. a) We have that

$$a_{n+1}^2 = a_n^2 + 1 + \frac{1}{4a_n^2}.$$

In particular, $a_{n+1}^2 > a_n^2 + 1$ and it follows by induction that $a_n^2 \geq n$.

We shall prove the other inequality by using induction again. It is obvious for $n = 1$. Suppose that $a_n^2 < n + \sqrt[3]{n}$. Then $a_{n+1}^2 < n + \sqrt[3]{n} + 1 + \dfrac{1}{4n}$ and it is enough to check that

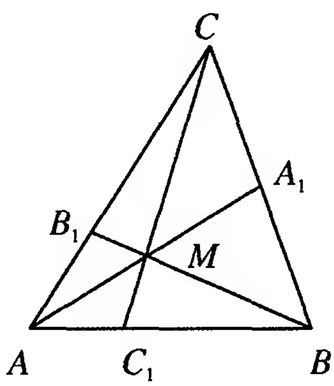
$$\begin{aligned}\sqrt[3]{n} + \frac{1}{4n} < \sqrt[3]{n+1} \quad &\Longleftrightarrow \quad \frac{1}{4n} < \sqrt[3]{n+1} - \sqrt[3]{n}\\ &\Longleftrightarrow \quad \sqrt[3]{n^2} + \sqrt[3]{n(n+1)} + \sqrt[3]{(n+1)^2} < 4n\\ &\Longleftrightarrow \quad 1 + \sqrt[3]{1 + \frac{1}{n}} + \sqrt[3]{\left(1 + \frac{1}{n}\right)^2} < 4\sqrt[3]{n}.\end{aligned}$$

This inequality follows by the inequalities $1 + \dfrac{1}{n} \le 2$ and $1 + \sqrt[3]{2} + \sqrt[3]{4} = \dfrac{1}{\sqrt[3]{2} - 1} < 4$.

b) The statement is a consequence of the inequalities

$$0 \le a_n - \sqrt{n} < \sqrt{n + \sqrt[3]{n}} - \sqrt{n} = \frac{\sqrt[3]{n}}{\sqrt{n + \sqrt[3]{n}} + \sqrt{n}} < \frac{\sqrt[3]{n}}{\sqrt{n}} = \frac{1}{\sqrt[6]{n}}.$$

11.2. Let A_1 be the midpoint of the segment BC. Then Ceva's theorem implies that



$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1,$$

i.e., $\dfrac{AC_1}{C_1B} = \dfrac{B_1A}{B_1C}$. Hence we have $B_1C_1 \parallel BC$, i.e. $S_{BC_1M} = S_{CB_1M} = 2S_{AC_1M}$ and we get that $S_{AB_1M} = S_{AC_1M}$. Then

$$\frac{1}{3} = \frac{S_{AC_1M}}{S_{AMC}} = \frac{C_1M}{MC} = \frac{S_{BC_1M}}{S_{BMC}} = \frac{2S_{AC_1M}}{2S_{BA_1M}}$$

and therefore $S_{BA_1M} = 3S_{AC_1M}$.

Conversely, let $S_{AC_1M} = 1$, $S_{CB_1M} = 2$, $S_{BA_1M} = 3$, $S_{BC_1M} = x$, $S_{CA_1M} = 3y$ and $S_{AB_1M} = 2z$. We have to show that $y = 1$. Note that

$$\frac{1}{2(z+1)} = \frac{S_{AC_1M}}{S_{AMC}} = \frac{C_1M}{CM} = \frac{S_{C_1MB}}{S_{CMB}} = \frac{x}{3(y+1)}.$$

Analogously,

$$\frac{3}{x+1} = \frac{3y}{2(z+1)} \quad \text{and} \quad \frac{2}{3(y+1)} = \frac{2z}{x+1}.$$

Multiplying these equalities gives $xyz = 1$. Hence $z = \dfrac{1}{xy}$ and the first equality implies that

$$xy = \frac{3y^2 + 3y - 2}{2}. \tag{1}$$

Analogously, the second equality gives

$$2\left(1 + \frac{1}{xy}\right) = xy + y. \tag{2}$$

Plugging (1) in (2) leads to

$$(3y^2 + 3y - 2)^2 + 2y(3y^2 + 3y - 2) - 12y(y + 1) = 0,$$

i.e.,

$$(y - 1)(3(y + 2)(3y^2 + 3y + 2) + 6y^2 - 16) = 0.$$

It follows by (1) that $3y^2 + 3y > 2$ and since $y > 0$ we get $3(y + 2)(3y^2 + 3y + 2) + 6y^2 - 16 > 6(3y^2 + 3y + 2) - 16 > 8$. Thus $y = 1$, $x = 2$ and $z = \dfrac{1}{2}$ which completes the solution.

11.3. We shall prove that Elitza has a winning strategy. If the polynomial is $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ and Alexander writes a_0, a_1, a_2 or a_3, then Elitza writes respectively $a_1 = a_0$, $a_0 = a_1$, $a_3 = a_2$ or $a_2 = a_3$; if he writes a_4, she writes $a_1 = 1$.

In a similar way Elitza is able to get $a_1 \leq a_0$ and $a_3 \leq a_2$ after her second move. Suppose that the polynomial obtained has an integer root $-y$. Then $y \geq 1$ and hence $a_4 = y^3(a_1 - a_0y) + a_3 - a_2y \leq 0$, which is a contradiction.

12.1. a) Since $\lim\limits_{x\to+\infty} f'(x) = +\infty$ and $\lim\limits_{x\to-\infty} f'(x) = -\infty$, it is enough to show that the local minimum m of $f'(x)$ is positive. Since the equation $f''(x) = 0$ has two real roots $x_1 > x_2$, it follows that $m = f'(x_1) > 0$. Now it is easy to check that $x_1 \in (-1; 0)$ and $m > 0$.

b) It follows from a) that the equation $f'(x) = 0$ has a unique real root. Since $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = +\infty$ and $f(0) < 0$, we conclude that the equation $f(x) = 0$ has exactly two real roots. To find them, set $y = 2003$ and consider $f(x) = 0$ as a quadratic equation with respect to y. We have

$$y_{1,2} = \frac{x \pm x(4x + 3)}{2},$$

and then either $2y = x - x(4x + 3)$ or $2y = x + x(4x + 3)$. For $y = 2003$ the first equation has no real roots and the second one has two real roots $x_{1,2} = \dfrac{-1 \pm \sqrt{4007}}{2}$.

12.2. Set $PT \cap BC = P_1$, $NT \cap AB = N_1$ and $MT \cap AC = M_1$. The triangles N_1MT, PTM_1 and TP_1N are similar to $\triangle ABC$. Set $k_1 = \frac{N_1M}{AB}$, $k_2 = \frac{PT}{AB}$ and $k_3 = \frac{TP_1}{AB}$. Then

$$(1) \qquad k_1 + k_2 + k_3 = 1,$$

since

$$k_1 + k_2 + k_3 = \frac{N_1M}{AB} + \frac{PT}{AB} + \frac{TP_1}{AB} = \frac{N_1M}{AB} + \frac{AN_1}{AB} + \frac{MB}{AB} = 1.$$

a) It is clear that $\frac{AM}{MB} = \frac{PT + N_1M}{AB} = \frac{k_1 + k_2}{k_3}$. Analogously, $\frac{BN}{NC} = \frac{k_1 + k_3}{k_2}$, $\frac{CP}{PA} = \frac{k_2 + k_3}{k_1}$. It follows by $\frac{AM}{MB} = \frac{BN}{NC}$ that $\frac{k_1 + k_2}{k_3} = \frac{k_1 + k_3}{k_2}$, i.e., $(k_2 - k_3)(k_1 + k_2 + k_3) = 0$. Hence $k_2 = k_3$. We get in the same way that $k_1 = k_2$ and then $k_1 = k_2 = k_3$. Hence $PT = TP_1$ and since $PP_1 || AB$, it follows that the line CT meets AB at its midpoint. Analogously, the lines BT and AT meet AC and BC at their midpoints. Hence T is the centroid of $\triangle ABC$.

b) We have

$$\begin{aligned} S_{MNP} &= S_{MNT} + S_{NPT} + S_{PMT} = S_{MBT} + S_{TNC} + S_{PAT} \\ &= \frac{1}{2}(S_{MBP_1T} + S_{TNCM_1} + S_{PAN_1T}) = \frac{S_{ABC}}{2}(1 - k_1^2 - k_2^2 - k_3^2). \end{aligned}$$

It follows by the inequality $k_1^2 + k_2^2 + k_3^2 \geq \frac{(k_1 + k_2 + k_3)^2}{3}$ and (1) that $k_1^2 + k_2^2 + k_3^2 \geq \frac{1}{3}$. Then

$$S_{MNP} \leq \frac{S_{ABC}}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}S_{ABC}.$$

12.3. Denote by A, B and C the three familiar people in the group.

Let $n = 2k + 1$ be odd integer. Then any of A, B and C has at least $k + 1$ familiar ($k - 1$ of them are not A, B or C). Denote by T the set of all people except A, B and C a let a_i, $i = 0, 1, 2, 3$, be the set of the people in T who have exactly i familiar among A, B and C.

Then $a_0 + a_1 + a_2 + a_3$ is the number of all members of T, i.e. we have

$$a_0 + a_1 + a_2 + a_3 = 2k - 2.$$

On the other hand, $a_1 + 2a_2 + 3a_3$ is the number of all familiar to A, B and C, i.e. we have

$$a_1 + 2a_2 + 3a_3 \geq 3k - 3.$$

Hence

$$\begin{aligned} 3k - 3 &\leq a_1 + 2a_2 + 3a_3 = a_0 + a_1 + a_2 + a_3 + a_2 + 2a_3 \\ &= 2k - 2 + a_2 + 2a_3 \end{aligned}$$

and therefore $a_2 + 2a_3 \geq k - 1$.

Since any familiar to two of A, B and C is a member of a triple of familiar people and any familiar to A, B and C is member of three such triples, then the number of these triples is at least $1 + a_2 + 3a_3$. Thus $1 + a_2 + 3a_3 > a_2 + 2a_3 \geq k - 1$, which means that the number of the triples is not less than k.

It remains to construct an example with k triples of familiar people. Let there are no familiar people in T. If A is familiar to exactly $k - 1$ people of T, and B and C – to the remaining $k - 1$, then the number of the triples is k.

Let $n = 2k$ be even number. As in the previous case , we get that the number of the triples of familiar people is at least $k + 1$. If A and B have exactly one common familiar person from T (it is possible, since $|T| = 2k - 3$ and the familiar to A and B are at least $k - 1$) who is not familiar to C, then the number of the triples is exactly $k + 1$.

So the answer of the problem is k for $n = 2k + 1$ and $k + 1$ for $n = 2k$.

Remark. The problem can be solved applying of the inclusion-exclusion principle to the sets of persons which are familiar to A, B and C, respectively.

Spring Mathematical Competition

8.1. a) The answer is *no*. Suppose that the numbers written in the vertices of the octagon are $a_1, a_2, \dots, a_8$ and

$$\begin{aligned} &a_1 + a_2 + a_3 \geq 14, \\ &a_2 + a_3 + a_4 \geq 14, \\ &\dots\dots\dots \\ &a_7 + a_8 + a_1 \geq 14, \\ &a_8 + a_1 + a_2 \geq 14. \end{aligned}$$

Summing up these inequalities gives

$$3(a_1 + a_2 + \cdots + a_8) \geq 8.14 = 112.$$

On the other hand, $a_1 + a_2 + \cdots + a_8 = 1 + 2 + \cdots + 8 = 36$. Hence $108 \geq 112$, a contradiction.

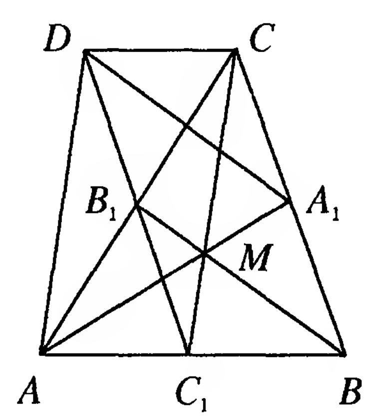
b) The answer is *yes* as the following example shows:

$$a_1 = 1, \ a_2 = 5, \ a_3 = 6, \ a_4 = 2, \ a_5 = 4, \ a_6 = 7, \ a_7 = 3, \ a_8 = 8.$$

c) The answer is *no*. Assume the contrary. The numbers 2 and 3 cannot be at adjacent vertices; otherwise, the numbers written on the left and on the right of them must be at lest 8 which is impossible. On the other hand, we may assume that $a_1 = 1$. Then it is easy to see that $a_4 = 2$ and $a_6 = 3$ or $a_4 = 3$ and $a_6 = 2$, and hence $a_5 = 8$. Now, considering the four cases for the vertex, where 4 is written, we see that the number 8 must appear again, which is a contradiction.

8.2. Since $A_1D \parallel MB_1$ and $A, B_1, M, C_1 \in k$, it follows that

$$\sphericalangle AA_1D = \sphericalangle AMB_1 = \sphericalangle AC_1B_1 = \sphericalangle ABC.$$



Using that $B_1C_1 \parallel BA_1$ and $A_1D \parallel BB_1$ we conclude that BA_1DB_1 is a parallelogram. Hence $BA_1 = B_1D = B_1C_1$. On the other hand, $AB_1 = B_1C$ and therefore AC_1CD is a parallelogram. In particular, $AD \parallel CC_1$ and then

$$\begin{aligned} \sphericalangle DAA_1 &= \sphericalangle CMA_1 = \sphericalangle AMC_1 \\ &= \sphericalangle AB_1C_1 = \sphericalangle ACB. \end{aligned}$$

Therefore

$$\sphericalangle ADA_1 = 180^\circ - \sphericalangle DAA_1 - \sphericalangle DA_1A = 180^\circ - \sphericalangle ABC - \sphericalangle ACB = \sphericalangle CAB.$$

8.3. Set $m = 2^k p$, where $p > 1$ is an odd integer. Then

$$2003^m - 1 = 2003^{2^k p} - 1 = (2003^{2^k} - 1)K,$$

where K is a sum of p even integers and 1; in particular, K is odd. Hence m must be of the form $m = 2^k$. In this case we have that

$$\begin{aligned} 2003^{2^k} - 1 &= (2003^{2^{k-1}} - 1)(2003^{2^{k-1}} + 1) \\ &= (2003^{2^{k-1}} + 1)(2003^{2^{k-2}} + 1)\dots(2003 + 1)(2003 - 1). \end{aligned}$$

Since $2003^{2^i} + 1 \equiv 2 \pmod 4$, then 2^{k+2} divides $2003^{2^k} - 1$ but 2^{k+3} does not (use that $2003 + 1 \equiv 4 \pmod 8$ and $2003 - 1 \equiv 2 \pmod 4$). Therefore $k + 2 = 2000$, i.e. $k = 1998$. Thus the desired number is equal to 2^{1998}.

9.1. If $x \neq -1$ and $y \neq -1$, we easily get that $y = a$. Plugging it in the second equation gives

$$(*) \qquad ax^2 - (a^2 - 2a - 1)x + a^3 = 0.$$

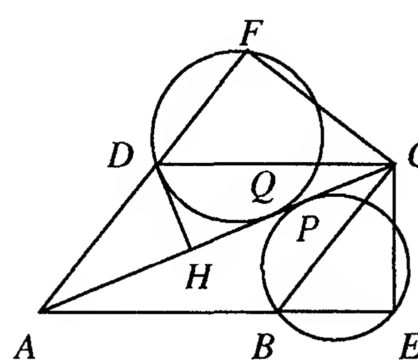
If $a = 0$ the system has a unique solution $(0;0)$. If $a \neq 0$, we consider the following two cases.

Case 1. -1 is a root of $(*)$. Then $(*)$ gives that $a = 1$ or -1. In both cases the system has no a solution.

Case 2. The equation $(*)$ has a double root. Then $a = -1$, $a = 1$ or $a = -\frac{1}{3}$. In the first two cases we get that $y = -1$ or $x = -1$, i.e., the system has no a solutions. If $a = -\frac{1}{3}$ the system has a unique solution $(x;y) = \left(\frac{1}{3}; -\frac{1}{3}\right)$.

Thus the desired values of a are 0 and $-\frac{1}{3}$.

9.2. Let $DH \perp AC$ ($H \in AC$). Then $\triangle AHD \sim \triangle AFC$ and $\triangle CHD \sim \triangle AEC$. Hence



$$\begin{aligned} AC^2 &= AH.AC + HC.AC \\ &= AF.AD + AE.CD \\ &= AQ^2 + AE.AB = AQ^2 + AP^2. \end{aligned}$$

Setting $QP = PC = x$, we get the equation

$$(1 + 2x)^2 = 1 + (1 + x)^2 \iff 3x^2 + 2x - 1 = 0,$$

which has a unique positive root $x = \frac{1}{3}$. Therefore $AC = 1 + 2x = \frac{5}{3}$.

9.3. We shall say that the order of Spas is 1, the order of his parents is 2, the order of their parents is 3, etc. Write the number of the heads of any dragon in ternary base: Spas has 1 head, his mother – 10 heads, his father 11 heads , etc. It follows by induction on the order of the dragons that the heads of the dragons of order n are n-digits numbers which ternary representation contains no the digit 2. It follows that if two dragons have $a + b = k$ ($a > b$) heads in total and k is written in ternary base, then a and b have 0 at the

positions, where k has 0 and 1 at the positions, where k has 2. Moreover, if k has 1 at some position, then one of a and b has 1 and the other one has 0 at the same position. So if the ternary representation of k contains no 1, then $a = b$ and hence k is not a good number. If this representation contains at least two 1's ($k = ...1...1...$), we have two possibilities: ($a = ...1...1...$, $b = ...0...0...$) and ($a = ...1...0...$, $b = ...0...1...$). Such a k is not a good number except for the case when k contains exactly two 1's and the other digits are 0.

If k contains exactly one 1, then a has 1 and b has 0 at this position and the other digits of a and b are uniquely determined. The only exception is the case when the remaining digits of k are 0. All the numbers with one 1 and at least one 2 are good.

Since $2003 = 2202012_3$, the number 2003 is good. Let us count all good numbers with at most 7 digits. One has $\binom{7}{2} = 21$ numbers with exactly two 1's and 0 at the other positions. There are $2^6 - 1 = 63$ non-zero numbers with at most 6 digits equal to 0 or 2. There exist 7 possibilities to put 1 in such numbers. So we get $7 \cdot 63 = 441$ numbers and adding the above 21 numbers, we obtain 462 good numbers with at most 7 digits.

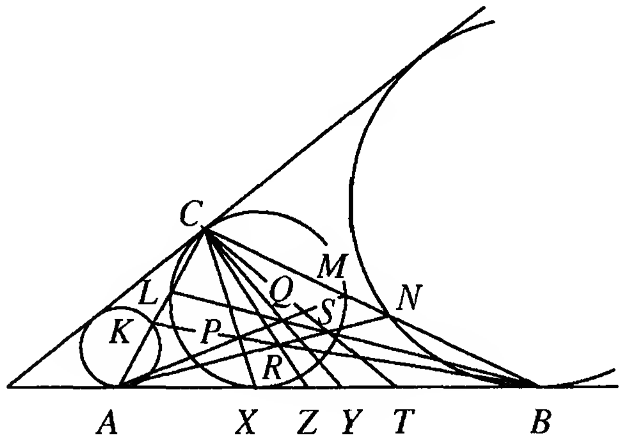
We shall count the 7-digit good numbers greater than 2202012_3. They are 2202021_3, 2202100_3, 2202102_3, 2202120_3, 2202122_3, 2202201_3, 2202210_3, 2202212_3, 2202221_3, 16 numbers of the form $\overline{221mnpq}_3$, where $m, n, p, q \in \{0; 2\}$ and $4 \cdot 8 = 32$ numbers of the form $\overline{222mnpq}_3$, where exactly one of the digit m, n, p, q is equal to 1. Thus, there are $1 + 9 + 16 + 32 = 58$ 7-digit good numbers greater than 2202012_3. Hence we have $462 - 58 = 404$ good numbers smaller than 2003.

10.1. a) Set $t = \dfrac{x^2}{x-1}$. Then $x^2 - tx + t = 0$. This quadratic equation has a real root if its discriminant is non-negative. Then $t^2 - 4t \geq 0$ which shows that the desired range is $(-\infty; 0] \cup [4; +\infty)$.

b) Write the equation in the form $f(x) = x^4 - ax^2(x-1) + (x-1)^2 = 0$. Obviously $x = 1$ is not a solution. Then dividing both sides by $(x-1)^2$ and setting $t = \dfrac{x^2}{x-1}$, we get the equation $t^2 - at + 1 = 0$. If its discriminant $a^2 - 4$ is negative, i.e. $a \in (-2; 2)$, the equation $f(x) = 0$ has no real solutions. If $a \in (-\infty; -2] \cup [2; +\infty)$, denote by t_1 and t_2 the roots of the above equation.

Now a) implies that $f(x) = 0$ has no real roots if and only if $t_1, t_2 \in (0; 4)$. Setting $g(t) = t^2 - at + 1$, this is equivalent to $\left| \begin{array}{l} g(0) > 0 \\ g(4) > 0 \\ 0 < \dfrac{a}{2} < 4 \end{array} \right.$ which gives $a \in \left(0; \dfrac{17}{4}\right)$. Having in mind that $a \in (-\infty; -2] \cup [2; +\infty)$, we get that $a \in \left[2; \dfrac{17}{4}\right)$.

10.2. If E and F are the second tangent points of k_1 and k_2 with the arms of the angle, then the equalities $AF^2 = AL.AC$, $CE^2 = CK.CA$ and $AF = CE$ imply that $AL = CK$. Hence $AK = CL$ and analogously $CM = BN$. On the



other hand, Ceva's theorem gives

$$\frac{AX}{XB} = \frac{AK.CM}{KC.MB} \quad \text{and} \quad \frac{AT}{TB} = \frac{AL.CN}{LC.NB}.$$

Multiplying these equalities gives $\dfrac{AX}{XB} = \dfrac{TB}{AT}$ and therefore

$$\frac{AX + XB}{XB} = \frac{TB + AT}{AT} \iff AT = BX \iff AX = BT.$$

Analogously $AZ = YB$ which implies that $XZ = YT$.

10.3. a) We shall prove that 4 measurements are enough. Denote the balls by $1, 2, 3, 4, 5, 6$ and measure consecutively $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ and $\{1, 5\}$.

Case 1. If all the measurements show radioactivity, then 1 is a radioactive ball. If $\{1, a\}$, $a = 2, 3, 4, 5$, contains two radioactive balls, then a is radioactive; otherwise, it is not. Hence we know which of the balls $1, 2, 3, 4, 5$ are radioactive and hence we also know whether 6 is radioactive or not.

Case 2. If some of the measurements shows no radioactivity, then 1 is not a radioactive ball. Hence we again know which of the balls $1, 2, 3, 4, 6$ are radioactive and then whether 6 is radioactive or not.

Assume that $L(6) \leq 3$, i.e, three measurements are enough. After two measurements we have $3^2 = 9$ possibilities. After the first measurement $1, 2, 3, 4, 5$ or 6 balls can be chosen. It is clear that among any 5 or 6 balls there are at least 2 radioactive; so a measurement with 5 or 6 gives no information.

Let x, $x \leq 4$, balls are chosen in the first measurement. If $x = 1$ and the answer is "one radioactive ball", then the possibilities for the radioactive balls are $\dbinom{5}{2} = 10$. On the other hand, the number of the possibilities for the other two measurements are $3^2 = 9$. Analogously, for a first measurement of:

– two balls and answer "one radioactive ball", the number of the possibilities is $\binom{2}{1}\binom{4}{2} = 12 > 9$;

– three balls and answer "more than one radioactive ball", the number of the possibilities is $\binom{3}{2}\binom{3}{1} + \binom{3}{3} = 10 > 9$;

– three balls and answer "more than one radioactive ball", the number of the possibilities is $\binom{4}{2}\binom{2}{1} + \binom{4}{3} = 13 > 9$.

Thus three measurements are not enough and so $L(6) = 4$.

b) We shall show that $\left[\frac{n+5}{2}\right]$ measurements are enough for finding the three radioactive balls which will imply the desired inequality. Let $n = 2t - \varepsilon$, where $\varepsilon \in \{0, 1\}$. Let us either pair the balls (if $\varepsilon = 0$) or take a ball and pair the remaining (if $\varepsilon = 1$). In both case we check any pair and the taken ball (if $\varepsilon = 1$) for radioactivity. Two cases are possible.

1. There is one set with two radioactive balls and one set with one radioactive balls

2. There are three sets with one radioactive ball.

In both cases three measurements are enough to find the radioactive balls. The total numbers of the measurements is $t - 1 + 3 = t + 2$. Since

$$\left[\frac{n+5}{2}\right] = \left[\frac{2t-\varepsilon+5}{2}\right] = t + 2 + \left[\frac{1-\varepsilon}{2}\right] = t + 2,$$

we get that $L(n) \leq \left[\frac{n+5}{2}\right]$.

11.1. If $a \geq 2$, then the roots x_1 and x_2 of the equation $x^2 - ax + 1 = 0$ are positive and $x_1x_2 = 1$. In particular, $S_n > 0$ for $n = 1, 2, \ldots$.

a) We have

$$\begin{aligned}
\frac{S_{n-1}}{S_n} \geq \frac{S_n}{S_{n+1}} \quad &\Longleftrightarrow \quad (x_1^{n-1} + x_2^{n-1})(x_1^{n+1} + x_2^{n+1}) \geq (x_1^n + x_2^n)^2 \\
&\Longleftrightarrow \quad x_1^{n-1}x_2^{n+1} + x_2^{n-1}x_1^{n+1} \geq 2x_1^n x_2^n \\
&\Longleftrightarrow \quad (x_1x_2)^{n-1}(x_1 - x_2)^2 \geq 0,
\end{aligned}$$

which obviously holds.

b) Let $a \geq 2$ have the desired property. Then a) implies that

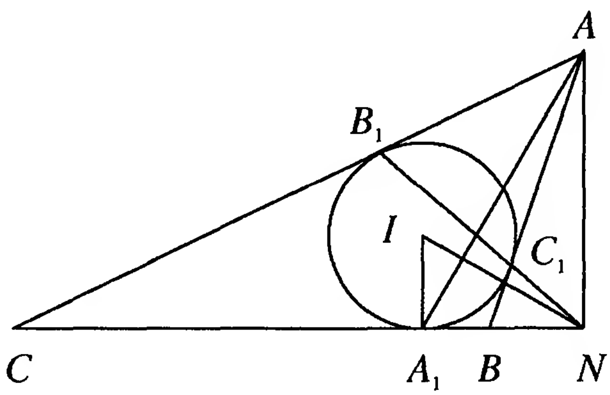
$$n\frac{S_1}{S_2} \geq \frac{S_1}{S_2} + \cdots + \frac{S_n}{S_{n+1}} > n - 1,$$

i.e., $\frac{S_1}{S_2} > 1 - \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the last inequality gives $\frac{S_1}{S_2} \geq 1$. Using Vieta's formulas we get $S_1 = a$, $S_2 = a^2 - 2$ and therefore $\frac{a}{a^2-2} \geq 1 \Longleftrightarrow \frac{(a+1)(a-2)}{a^2-2} \leq 0$. Since $a \geq 2$ we get $a = 2$.

Conversely, if $a = 2$, then $x_1 = x_2 = 1$ and $S_n = 2$ for any $n = 1, 2, \dots$. Hence

$$\frac{S_1}{S_2} + \frac{S_2}{S_3} + \cdots + \frac{S_n}{S_{n+1}} = n > n - 1.$$

11.2. We shall use the standard notation for the elements of $\triangle ABC$. We may assume that $b > c$. Denote by I the incenter of $\triangle ABC$. Then the condition $A_1N = r\sqrt{3}$ implies that INA_1 is a right-angled triangle with $\sphericalangle NIA_1 = 60°$. We shall prove that $AA_1 \perp IN$. If so, then $\sphericalangle AA_1N = 60°$. Now, if M is the midpoint of the segment AA_1, then $\triangle MNA_1$ is equilateral and therefore $\triangle ANM$ is isosceles with $\sphericalangle ANM = \sphericalangle MAN = 30°$. Thus $\sphericalangle ANC = 90°$.



To prove that $AA_1 \perp IN$, note that this is equivalent to the equality $AI^2 - IA_1^2 = AN^2 - A_1N^2$. The Cosine theorem for $\triangle ANB$ gives

$$AN^2 = c^2 + BN^2 + 2c.BN\cos\beta = c^2 + BN^2 + 2c.BN\frac{a^2 + c^2 - b^2}{2ac}.$$

On the other hand, the Menelaus theorem for $\triangle ABC$ and the line B_1C_1 implies that

$$\frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} \cdot \frac{BN}{NC} = 1.$$

Hence $\dfrac{p-c}{p-b} \cdot \dfrac{BN}{BN+a} = 1$, i.e., $BN = \dfrac{a(p-b)}{b-c}$, where p is the semiperimeter of $\triangle ABC$. Then $A_1N = BN + p - b$ and therefore

$$\begin{aligned} AN^2 - A_1N^2 &= c^2 - (p-b)^2 + BN\frac{a^2 + c^2 - b^2 - a^2 - ac + ab}{a} \\ &= c^2 - (p-b)^2 - \frac{a(p-b)}{b-c} \cdot \frac{2(b-c)(p-a)}{a} \\ &= c^2 - (p-b)^2 - 2(p-a)(p-b) = (p-a)^2 \\ &= AI^2 - IC_1^2 = AI^2 - IA_1^2 \end{aligned}$$

which completes the proof.

Remark. The fact that $AA_1 \perp IN$ holds true for any $\triangle ABC$. It can be proved as above or using complex numbers or inversion. Note also that if P is

the second common point of the line AA_1 and the incircle of $\triangle ABC$, then the line NP is tangent to this circle. The data of our problem give $P \equiv M$.

11.3. We shall prove that $n = 6$. If we take 6 points in general position (no three are collinear), then the lines are 15 and any point lies on 5 lines, i.e. $n = 6$ is a solution of the problem.

Denote by k the number of the lines defined by the given n points. Assume that there is a line l containing 4 of the given points. Any of the points belongs to $\frac{k}{3} - 1$ lines different from l which means that there are at least $4\left(\frac{k}{3} - 1\right) + 1$ lines. Then

$$4\left(\frac{k}{3} - 1\right) + 1 \leq k,$$

i.e., $k \leq 9$. On the other hand, any point lying no on l belongs to at least four lines (the lines through the point and the four points on l) and hence $k \geq 12$, a contradiction. So any line contains at most 3 points. Let a of the lines contain 2 points. Then each of the other $k - a$ lines contains 3 points.

The number of the points (any of them counted $\frac{k}{3}$ times) is equal to $2a + 3(k - a)$ and then $2a + 3(k - a) = \frac{nk}{3}$. On the other hand, since n points define $\frac{n(n-1)}{2}$ lines (some of them may coincide) and any line containing three points is counted three times, then $a + 3(k - a) = \frac{n(n-1)}{2}$. Thus

$$a = \frac{n(n-1)(9-n)}{2(2n-9)}, \quad k = \frac{3n(n-1)}{2(2n-9)}.$$

Since $k \leq \frac{n(n-1)}{2}$, then $2n - 9 \geq 3$, i.e., $n \geq 6$. Now $a \geq 0$ implies that $n \leq 9$. For $n = 7$ the values of a and k are not integers and hence $n = 8$ or $n = 9$. For $n = 8$ one has that $k = 12, a = 4$ and for $n = 9$ we get that $k = 12, a = 0$.

Denote by l the maximal number of points in general position among the given n points. Then the remaining points belong to lines defined by these l points.

Case 1. Let $l = 3$ and let the respective points be A_1, A_2, A_3. Any of the other points lies on one of the lines A_1A_2, A_1A_3 and A_2A_3. Since any line contains at most 3 points, then we have at most 6 points, a contradiction.

Case 2. Let $l = 4$ and let the respective points be A_1, A_2, A_3, A_4. Since the total number of the points is at least 8, we may find a point belonging to exactly one of the lines defined by A_1, A_2, A_3, A_4. We may assume that the point is A_5 and $A_5 \in A_1A_2$. Then the points A_3, A_4, A_1, A_5 as well as A_3, A_4, A_2, A_5 are in general position. Hence all the points must belong to the lines defined by A_1, A_2, A_3, A_4; A_3, A_4, A_1, A_5 and A_3, A_4, A_2, A_5. The only common lines are A_3A_4 and $A_1A_2A_5$, i.e., all the points lie on two lines. This is a contradiction to the fact that any line contains at most 3 points.

Case 3. Let $l = 5$ and let the respective points be A_1, A_2, A_3, A_4, A_5. Any of the points A_1, A_2, A_3, A_4, A_5 belongs to exactly 4 lines. This means that A_6A_i, $i = 1, 2, \dots, 5$, is one of these lines. We may assume that $A_6 \in A_1A_2$. Then A_6A_3 is one of the lines A_3A_4, A_3A_5 or A_4A_5. Let us have, for example, $A_6 \in A_3A_4$. Then A_6A_5 is a new line, a contradiction.

12.1. a) We have

$$\begin{aligned} g'(x) &= \frac{(1-\cos x)[k(1+\cos x+\cos^2 x)-\cos^2 x]}{\cos^2 x} \\ &= \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{\cos^2 x}[k-f(x)] \\ &= \frac{(1-\cos x)[k-f(x)]}{f(x)}. \end{aligned}$$

b) Set $\cos x = t$. Then $t \in (0; 1]$ for $x \in \left[0; \frac{\pi}{2}\right)$ and $f(x) = h(t) = \frac{t^2}{1+t+t^2}$. Since $h'(t) = \frac{t^2+2t}{(1+t+t^2)^2}$, we see that $h'(t) > 0$ for $t \in (0; 1]$ and therefore $h(t)$ increases in this interval. Having in mind that $h(0) = 0$ and $h(1) = \frac{1}{3}$, we conclude that the range of $h(t)$ for $t \in (0; 1]$, i.e., the range of $f(x)$ for $x \in \left[0; \frac{\pi}{2}\right)$, is the interval $\left(0; \frac{1}{3}\right]$.

c) If $k < 0$, then $\lim\limits_{x \to \frac{\pi}{2}, x < \frac{\pi}{2}} g(x) = -\infty$, i.e, any $k < 0$ is not a solution of the problem. It is also easy to see that $k = 0$ is not a solution of the problem.

It follows by b) that if $k \geq \frac{1}{3}$, then $g'(x) > 0$ for $x \in \left(0; \frac{\pi}{2}\right)$, i.e., the function $g(x)$ increases in this interval. Now $g(0) = 0$ implies that $g(x) \geq 0$ for $x \in \left[0; \frac{\pi}{2}\right)$.

If $k \in \left(0; \frac{1}{3}\right)$, then the equation $g'(x) = 0$ has a unique root x_0 in the interval $\left(0; \frac{\pi}{2}\right)$. The considerations above show that $g'(x) < 0$ for $x \in (0; x_0)$, i.e., $g(x)$ decreases in this interval. Then $g(0) = 0$ implies that the given inequality is not true.

Thus, the desired values of k are $k \in \left[\frac{1}{3}; \infty\right)$.

12.2. a) We shall use the standard notation for the elements of $\triangle ABC$. The Cosine theorem for $\triangle ABM$ gives

$$\cot \sphericalangle AMB = \frac{AM^2 + BM^2 - AB^2}{2AM.MB \sin \sphericalangle AMB} = \frac{a^2 + b^2 - 5c^2}{12 S_{ABC}}.$$

Since $\cot 2\gamma = \frac{2\cos^2\gamma - 1}{2\sin\gamma\cos\gamma}$, the condition $\sphericalangle AMB = 2\gamma$ becomes

$$
\begin{aligned}
& \frac{a^2+b^2-5c^2}{3ab} = 2\cos\gamma - \frac{1}{\cos\gamma} \\
\iff & \frac{a^2+b^2-5c^2}{3ab} - \frac{a^2+b^2-c^2}{ab} = -\frac{2ab}{a^2+b^2-c^2} \\
\iff & \frac{a^2+b^2+c^2}{3ab} = \frac{ab}{a^2+b^2-c^2} \\
\iff & (a^2+b^2)^2 - c^4 = 3a^2b^2 \\
\iff & c^4 = a^4+b^4-a^2b^2.
\end{aligned}
$$

b) It is easy to check that $a^4+b^4-a^2b^2 \geq (a^2+b^2-ab)^2$ for any $a, b > 0$. Then a) implies that $a^2+b^2-2ab\cos\gamma = c^2 \geq a^2+b^2-ab$. Hence $\cos\gamma \leq \frac{1}{2}$, i.e., $\gamma \geq 60°$.

Remark. The inequality in b) follows also by the fact that M lies on the segment with endpoints the circumcenter O and the orthocenter H of $\triangle ABC$. Indeed, since $\sphericalangle AMB = 2\gamma = \sphericalangle AOB$, then H does not lie in the interior of the circumcircle of $\triangle AOB$. Then the inequality $\gamma < 90°$ shows that the points O, H and C lie on the same side of the line AB. Hence $180° - \gamma = \sphericalangle AHB \leq \sphericalangle AOB = 2\gamma$, i.e. $\gamma \geq 60°$.

12.3. Setting $h(x) = f(x+1) - 1$, it is easy to see that the conditions for $f(x)$ are equivalent to $h(x) = ax$ for any $x \in [1,2)$ and $h(h(x)) = -2x$ for any $x \in \mathbb{R}$. Then $h(-2x) = h(h(h(x))) = -2h(x)$; in particular, $h(0) = 0$.

It follows by induction that $h(4^n x) = 4^n h(x)$ and hence $h(x) > 0$ for $x \in [4^n, 2.4^n)$, where n is an arbitrary integer. Since $0 > -2x = h(h(x)) = h(ax)$ for $x \in [1,2)$, then $[a, 2a) \subset [2.4^k, 4^{k+1})$ for some integer k. Therefore $a = 2.4^k$.

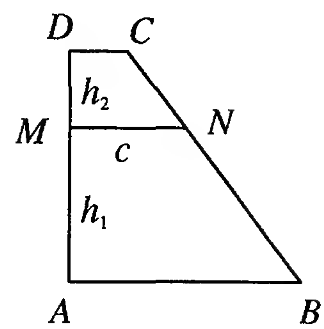
Conversely, if a has this form, then it is easy to check that the function

$$
h(x) = \begin{cases} ax, & x \in [4^n, 2.4^n), \\ -\dfrac{2x}{a}, & x \in [2.4^n, 4^{n+1}), \\ 0, & x = 0, \\ ax, & x \in (-4^{n+1}, -2.4^n], \\ -\dfrac{2x}{a}, & x \in (-2.4^n, -4^n], \end{cases}
$$

where n runs over all integers, has the desired properties. One can easily show that this is the only function with the above properties.

52. Bulgarian Mathematical Olympiad

Regional round



1. Let $ABCD$ be a right-angled trapezoid with area 10 and altitude $AD = 4$. Let the line $MN||AB$, $M \in AD$, $N \in BC$, divides it into two circumscribed trapezoids. Set $AB = a$, $CD = b$ $(a > b)$, $MN = c$, $AM = h_1$ and $DM = h_2$. Then $h_1^2 + (a-c)^2 = (a+c-h_1)^2$ and we get that $h_1 = \dfrac{2ac}{a+c}$.

Analogously $h_2 = \dfrac{2bc}{b+c}$ which implies that $\dfrac{h_1}{h_2} = \dfrac{a(b+c)}{b(a+c)}$. On the other hand, $\dfrac{h_1}{h_2} = \dfrac{a-c}{c-b}$ and therefore $(a+b)(ab-c^2) = 0$. Hence $c^2 = ab$ and we get that

$$h_1 = \frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}, \quad h_2 = \frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}.$$

Then $\sqrt{ab} = 2$ and $a+b = 5$, i.e., a and b are the roots of the equation $x^2 - 5x + 4 = 0$. Consequently $a = 4$, $b = 1$ and the radii equal $\dfrac{h_1}{2} = \dfrac{4}{3}$ and $\dfrac{h_2}{2} = \dfrac{2}{3}$.

2. For $n \leq 10$ Ann wins writing the numbers $1, 2, \ldots, 2^{n-1}$. Indeed, the result Ivo can get is a non-zero integer between -1023 and 1023, since it has the same sign as the largest remaining number $(2^j > 2^j - 1 = \sum_{i=0}^{j-1} 2^i)$.

For $n \geq 11$ the set C of Ann's numbers has $2^n - 1 > 2003$ different non-empty subsets. Hence the sums of numbers of two of them, say A and B, are congruent modulo 2003. If Ivo puts $+$ in front of the numbers of $A \setminus B$, $-$ in front of the numbers of $B \setminus A$ and deletes the remaining numbers of C, he wins.

3. It follows by the given condition that $4(an-1) < n + a(an)$ and $4an > n + a(an-1) - 1$, i.e.,

$$1 + a^2 - \frac{a+1}{n} < 4a < 1 + a^2 + \frac{4}{n}.$$

Letting $n \to \infty$ gives $1 + a^2 = 4a$, so $a = 2 - \sqrt{3}$ or $a = 2 + \sqrt{3}$.

The given equality for $n = 1$ yields that the first case is impossible.

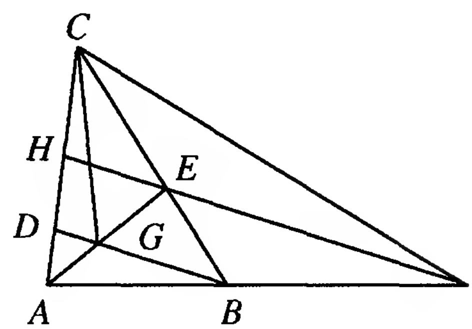
In the second case set $b = \left[\frac{n}{a}\right]$ and $c = \frac{n}{a} - b$. Since $a = 4 - \frac{1}{a}$, then

$$\begin{aligned}
n + [a[an]] &= \left[n + a\left[4n - \frac{n}{a}\right]\right] = [n + a(4n - b - 1)] \\
&= [a(4n - 1 + c)] = \left[\left(4 - \frac{1}{a}\right)(4n - 1 + c)\right] \\
&= \left[4(4n-1) - 4\left(\frac{n}{a} - c\right) + \frac{1-c}{a}\right] = 4(4n - 1 - b) \\
&= 4\left[4n - \frac{n}{a}\right] = 4[an].
\end{aligned}$$

Therefore $a = 2 + \sqrt{3}$ is the only solution of the problem.

4. Let $H = AC \cap EF$. Then $\measuredangle CDG = \measuredangle FHC$ and

$$\frac{CD}{DG} = \frac{BD}{DG} = \frac{FH}{HE} = \frac{FH}{HC}.$$



It follows that $\triangle CDG \sim \triangle FHC$ which implies that $\measuredangle GCD = \measuredangle CFH$. Hence

$$\begin{aligned}
\measuredangle BCG &= \measuredangle BCD - \measuredangle GCD \\
&= \measuredangle CEH - \measuredangle CFH = \measuredangle BCF.
\end{aligned}$$

5. Note first that the the triple $(0, 0, 0)$ is a solution of the system.

If $y = 0$, then it follows from the first equation that $x = -z$ and the second one gives that $x = z = 0$.

If $y \neq 0$, set $a = \frac{x}{y}$ and $b = \frac{z}{y}$. Then the system becomes

$$\left|\begin{array}{l} 1 + a + b = 3ay \\ 1 + a^2 + b^2 = 3ab \\ y(1 + a^3 + b^3) = 3b. \end{array}\right.$$

Hence $y = \frac{1 + a + b}{3a}$ and therefore

$$\left|\begin{array}{l} (1 + a + b)(1 + a^3 + b^3) = 9ab \\ 1 + a^2 + b^2 = 3ab. \end{array}\right.$$

Set $u = a + b$ and $v = ab$. Then

$$\left|\begin{array}{l} (1 + u)(1 + u^3 - 3uv) = 9v \\ 1 + u^2 - 2v = 3v, \end{array}\right.$$

Hence $v = \frac{u^2 + 1}{5}$ and plugging it into the first equation gives

$$0 = u^4 + u^3 - 6u^2 + u - 2 = (u - 2)(u^3 + 3u^2 + 1).$$

The case $u = 2$ leads to $v = 1$, $a = b = 1$ and gives the solution $(x, y, z) = (1, 1, 1)$. The function $f(u) = u^3 + 3u^2 + 1$ has a local maximum at $u = -2$ and a local minimum at $u = 0$. Since $f(0) = 1 > 0$, the equation $f(u) = 0$ has only one real root u_0 and $u_0 < -2$. Then $u_0^2 - 4\frac{u_0^2+1}{5} = \frac{u_0^2-4}{5} > 0$ which shows that the system

$$\left| \begin{array}{l} a + b = u_0 \\ ab = \dfrac{u_0^2+1}{5} \end{array} \right.$$

has two solution.

Thus the given system has four real solutions.

6. Let $p_1, p_2, \ldots, p_n$ be all prime divisors of all possible differences of two distinct numbers of C.

Suppose that for any p_i there exists an integer α_i such that $c \equiv \alpha_i \pmod{p_i}$ for at most one $c \in C$. It follows by the Chinese remainder theorem that there is an integer k such that $k \equiv p_i - \alpha_i \pmod{p_i}$ for any i. Then the condition of the problem implies that p_j divides $a + k$ and $b + k$ for some j and some $a, b \in C$. Then $a \equiv b \equiv \alpha_j \pmod{p_j}$, a contradiction.

We conclude that for some prime number p each reminder modulo p appears at least twice. Assuming that any remainder appears exactly twice we get that the sum of the elements of C equals

$$pr + 2(0 + 1 + \cdots + p - 1) = p(r + p - 1), \ \ r \geq 1.$$

This is a contradiction, since 2003 is a prime number. Hence some remainder appears at least three times. Removing an element of C giving this remainder we obtain a new good set C'. (Indeed, for any k we can find $a, b \in C'$, $a \neq b$, such that p divides $a + k$ and $b + k$.)

52. Bulgarian Mathematical Olympiad

Final round

1. If $x_1 = 2$ and $x_2 = x_3 = x_3 = x_5 = -1$, then all six sums of the form $x_1 + x_q + x_r$, $2 \leq q < r \leq 5$ are equal to 0 and therefore $n \geq 7$

Let now 7 sums of the form $x_p + x_q + x_r$, $1 \leq p < q < r \leq 5$, be equal to 0. Since $\frac{7.3}{5} > 4$, there exists x_i which occurs at least 5 times in these sums. We may assume that $i = 1$.

The sums of the form $x_1 + x_q + x_r$, $2 \leq q < r \leq 5$, are 6. Hence at most one of them is non-zero. So we may assume that

$$\begin{aligned} x_1 + x_2 + x_4 &= x_1 + x_2 + x_5 = x_1 + x_3 + x_4 \\ &= x_1 + x_3 + x_5 = x_1 + x_4 + x_5 = 0. \end{aligned}$$

It follows that $x_2 = x_3 = x_4 = x_5 = -\frac{x_1}{2}$ So all the 6 sums $x_1 + x_q + x_r$, $2 \leq q < r \leq 5$ equal 0. Since the 7-th sum which equals 0 has the form $x_p + x_q + x_r$, $2 \leq p < q < r \leq 5$, we get that $3x_p = 0$, which implies that $x_1 = x_2 = x_3 = x_4 = x_5 = 0$.

2. a) Set $\sphericalangle NPC = \varphi_1$ and $\sphericalangle MPC = \varphi_2$. Then

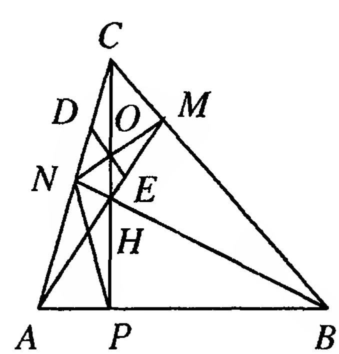
$$\frac{CN}{AN} = \frac{S_{NPC}}{S_{NPA}} = \frac{CP \sin \varphi_1}{AP \sin(90^\circ - \varphi_1)}$$

and hence

$$\tan \varphi_1 = \frac{CN.AP}{AN.CP}.$$

Analogously

$$\tan \varphi_2 = \frac{CM.BP}{BM.CP}.$$



Hence the equality $\varphi_1 = \varphi_2$ is equivalent to

$$\frac{CN.AP.BM}{AN.BP.CM} = 1$$

which follows from Ceva's theorem for $\triangle ABC$ and the lines AM, BN and CP.

b) Set $\sphericalangle NPC = \sphericalangle MPC = \varphi$, $\sphericalangle EPO = x$ and $\sphericalangle DPO = y$. It is easy to see that

$$x = y \iff \cot x = \cot y \iff \frac{\sin(\varphi - x)}{\sin x} = \frac{\sin(\varphi - y)}{\sin y}.$$

Assume that $E \in NH$ and $D \in CM$. Then

$$\frac{ME}{EH} = \frac{S_{MEP}}{S_{HEP}} = \frac{MP\sin(\varphi - x)}{PH\sin x} \iff \frac{\sin(\varphi - x)}{\sin x} = \frac{ME.PH}{EH.MP}.$$

Similarly $\dfrac{\sin(\varphi - y)}{\sin y} = \dfrac{DN.CP}{CD.PN}$. Hence using that PO is the bisector of $\sphericalangle NPM$, i.e., $\dfrac{PM}{PN} = \dfrac{MO}{NO}$ we have to prove that

$$\frac{ME}{EH} \cdot \frac{CD}{DN} \cdot \frac{PH}{CP} \cdot \frac{NO}{MO} = 1.$$

Set $\sphericalangle NOD = \delta$ and $\sphericalangle EOP = \psi$. Then

$$\frac{ME}{EH} = \frac{S_{MEO}}{S_{HEO}} = \frac{MO\sin\delta}{OH\sin\psi} \text{ and } \frac{CD}{DN} = \frac{S_{CDO}}{S_{DNO}} = \frac{CO\sin\psi}{ON\sin\delta}.$$

Hence it remains to show that $\dfrac{OC.PH}{OH.PC} = 1$.

Applying the Menelaus theorem for $\triangle BHC$ and the line MN, $\triangle CHM$ and the line AB, and $\triangle BHM$ and the line AC, we get that

$$\frac{BN.HO.CM}{NH.OC.MB} = 1, \quad \frac{CP.HA.MB}{PH.AM.BC} = 1 \text{ and } \frac{HN.BC.MA}{BN.CM.HA} = 1.$$

Multiplying these equalities gives the desired result.

3. Let k have the given property. We have that $y_3 = 2k - 2 = 4a^2$ $(a \geq 0)$, i.e., $k = 2a^2 + 1$, Further $y_4 = 8k^2 - 20k + 13$ and $y_5 = 32k^3 - 120k^2 + 148k - 59 = 256a^6 - 96a^4 + 8a^2 + 1$.

If $a = 0$ we get that $k = 1$ and the given sequence is $1, 1, 0, 1, 1, 0, \ldots$. Hence $k = 1$ is a solution of the problem.

Let $a > 0$. Its is easy to check that

$$(16a^3 - 3a)^2 \geq y_5 = 256a^6 - 96a^4 + 8a^2 + 1 > (16a^3 - 3a - 1)^2.$$

Since y_5 is a perfect square, the first inequality must be equality, i.e., $a = 1$ and then $k = 3$.

We shall prove that $k = 3$ is a solution of the problem. In this case the sequence is defined by $y_1 = y_2 = 1$ and $y_{n+2} = 7y_{n+1} - y_n - 2$ for $n \geq 1$. Since $y_3 = 2^2$, $y_4 = 5^2$ and $y_5 = 13^2$, it is natural to conjecture that $y_n = u_{2n-3}^2$ for $n \geq 2$ where $\{u_n\}_{n=1}^{\infty}$ is the Fibonacci sequence: $u_1 = u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$. To prove this, we first note that $u_{n+2} = 3u_n - u_{n-2}$ and $u_{n+2}u_{n-2} - u_n^2 = 1$ for any odd $n \geq 3$. It follows that $(u_{n+2} + u_{n-2})^2 = 9u_n^2$ and thus

$$u_{n+2}^2 = 9u_n^2 - u_{n-2}^2 - 2u_{n-2}u_{n+2} = 7u_n^2 - u_{n-2}^2 - 2.$$

Hence $y_n = u_{2n-3}^2$ for $n \geq 2$.

4. Let $A = \{a_1, a_2, \dots, a_n\}$ be a uniform set. Set $S = a_1 + a_2 + \cdots + a_n$. It follows from the given condition that $S - a_i$ is an even number for any $i = 1, 2, \dots, n$ Suppose that the number S is even. Then all the numbers a_i are even. Set $a_i = 2b_i$, $i = 1, 2, \dots, n$. Then it is easy to see that the set $B = \{b_1, b_2, \dots, b_n\}$ is uniform, too. So, we may assume that S is an odd number and whence $a_1, a_2, \dots, a_n$ and n are also odd numbers.

We shall prove that $n = 7$. It is not difficult to check that $\{1, 3, 5, 7, 9, 11, 13\}$ is a uniform set. It remains to show that there are no unform sets with 5 elements, since it is is obvious that the sets with 3 elements are not uniform.

Suppose that $A = \{a_1, a_2, a_3, a_4, a_5\}$ is a uniform set and let $a_1 < a_2 < a_3 < a_4 < a_5$. Considering the set $A \setminus \{a_1\}$ we see that either $a_2 + a_5 = a_3 + a_4$ or $a_2 + a_3 + a_4 = a_5$. Considering the set $A \setminus \{a_2\}$ we get that either $a_1 + a_5 = a_3 + a_4$ or $a_1 + a_3 + a_4 = a_5$.

- If $a_2 + a_5 = a_3 + a_4$ and $a_1 + a_5 = a_3 + a_4$, then $a_1 = a_2$.
- If $a_2 + a_5 = a_3 + a_4$ and $a_1 + a_3 + a_4 = a_5$, then $a_1 = -a_2$
- If $a_2 + a_3 + a_4 = a_5$ and $a_1 + a_5 = a_3 + a_4$, then $a_1 = -a_2$.
- If $a_2 + a_3 + a_4 = a_5$ and $a_1 + a_3 + a_4 = a_5$, then $a_1 = a_2$.

Since all the above possibilities lead to a contradiction, we conclude that there are no unform sets with 5 elements.

5. Let $a + b + c = a^2 + b^2 + c^2 = t$. Then $t \geq 0$. On the other hand, the Root mean square – Arithmetic mean inequality implies that

$$\frac{a^2 + b^2 + c^2}{3} \geq \frac{(a + b + c)^2}{9} \iff 3t \geq t^2.$$

Hence $t \in \{0, 1, 2, 3\}$. If $t = 0$ or $t = 3$, then $a = b = c = 0$ or $a = b = c = 1$, respectively, and the the statement is trivial.

Let $t = 1$. Denote by d the product of the denominators of $|a|$, $|b|$ and $|c|$. Then $x = ad$, $y = bd$ and $z = cd$ are integers for which $x + y + z = d$ and $x^2 + y^2 + z^2 = d^2$. We may assume that $z > 0$. Note that

$$(x + y + z)^2 = x^2 + y^2 + z^2 \iff xy + yz + xz = 0 \iff (x + z)(y + z) = z^2.$$

It follows that $x + z = rp^2$, $y + z = rq^2$ and $z = |r|pq$, where p and q are coprime positive integers, and r is a non-zero integer. Since $d = x + y + z = r(p^2 + q^2) - |r|pq > 0$ we get that $r > 0$. Then

$$a = \frac{x}{d} = \frac{p(p - q)}{p^2 + q^2 - pq}, \quad b = \frac{y}{d} = \frac{q(q - p)}{p^2 + q^2 - pq}, \quad c = \frac{z}{d} = \frac{pq}{p^2 + q^2 - pq}$$

and thus $abc = \dfrac{[pq(p - q)]^2}{(pq - p^2 - q^2)^3}$.

It remains to show that $pq(p - q)$ and $p^2 + q^2 - pq$ are coprime integers. Suppose that s is a prime divisor of $pq(p - q)$ and $p^2 + q^2 - pq$. Let $s|p$. Since $s|p^2 + q^2 - pq$, then $s|q$, a contradiction. Analogously, $s \nmid q$. Hence $s|p - q$. Then $s|(p - q)^2 - (p^2 + q^2 - pq) = pq$, which is impossible.

The case $t = 2$ can be reduced to the case $t = 1$ by setting $a_1 = 1 - a$, $b_1 = 1 - b$ and $c_1 = 1 - c$. Indeed, it is easy to check that $a_1 + b_1 + c_1 = a_1^2 + b_1^2 + c_1^2 = 1$ and $a_1 b_1 c_1 = -abc$.

6. Denote by m and a the degree and the leading coefficient of $P(x)$, respectively. Let x_n be an integer solution of the equation $P(x) = 2^n$. Since $\lim\limits_{n\to\infty} |x_n| = +\infty$, then $\lim\limits_{n\to\infty} \dfrac{a|x_n|^m}{2^n} = 1$ and hence $\lim\limits_{n\to\infty} \left|\dfrac{x_{n+1}}{x_n}\right| = \sqrt[m]{2}$.

On the other hand, $x_{n+1} - x_n$ divides $P(x_{n+1}) - P(x_n)$ and thus $|x_{n+1} - x_n| = 2^{k_n}$ for some $k_n \geq 0$. Then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{2^{k_n}}{|x_n|} + \varepsilon_n,$$

where $\varepsilon_n = \pm 1$ and we get that

$$\sqrt[m]{2} = \lim_{n\to\infty}\left(\frac{2^{k_n}}{|x_n|} + \varepsilon_n\right) = \lim_{n\to\infty}\left(2^{k_n}\sqrt[m]{\frac{a}{2^n}} + \varepsilon_n\right).$$

Note that ε_n equals either 1 or -1 for infinitely many n. Since the two cases are similar, we shall consider only the second one. Let $1 = \varepsilon_{i_1} = \varepsilon_{i_2} = \cdots$. Then

$$\sqrt[m]{2} + 1 = \sqrt[m]{a} \lim_{j\to\infty} 2^{k_{i_j} - i_j}$$

and hence the sequence of integers $k_{i_j} - i_j$ converges to some integer ℓ. It follows that $\left(\sqrt[m]{2} + 1\right)^m = a2^{m\ell}$ is a rational number. According to the Eisenstein criteria, the polynomial $x^m - 2$ is irreducible. Hence $(x-1)^m - 2$ is the minimal polynomial of $\sqrt[m]{2} + 1$. It follows that $(x-1)^m - 2 = x^m - a2^{m\ell}$ which is possible only for $m = 1$.

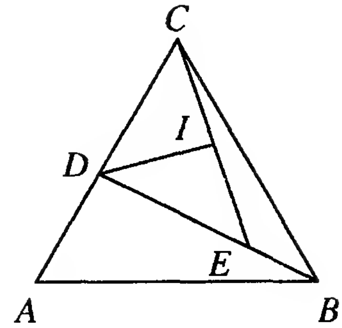
Let $P(x) = ax + b$. Then $a(x_2 - x_1)$ divides 2 and thus $a = \pm 1, \pm 2$. Now it follows easily that all polynomials with the desired property are of the form $P(x) = a(x + b)$, where $a = \pm 1, \pm 2$ and b is an arbitrary integer.

Team selection test for 20. BMO

1. Note that

$$\measuredangle EDC = 2\measuredangle CED \iff DI = EI,$$

where DI ($I \in CE$) is the bisector of $\measuredangle CDE$. Setting $BD = 2AD = 4BE = 4x$ and $AC = BC = y$ we get



$$DI = \frac{\sqrt{ED.CD((ED+CD)^2 - CE^2)}}{ED+CD}$$
$$= \frac{\sqrt{3x(y-2x)((x+y)^2 - CE^2)}}{x+y},$$
$$EI = CE\frac{ED}{ED+CD} = CE\frac{3x}{x+y}.$$

It is easy to deduce from here that

$$DI = EI \iff CE^2 = (x+y)(y-2x).$$

The last identity follows by Steward's theorem for $\triangle BCD$:

$$\begin{aligned} CE^2 &= \frac{BC^2DE + CD^2BE}{BD} - BE.DE \\ &= \frac{3y^2 + (y-2x)^2}{4} - 3x^2 = (x+y)(y-2x). \end{aligned}$$

2. We have

$$\begin{aligned} a+b+c-\frac{a}{b^2+1}-\frac{b}{c^2+1}-\frac{c}{a^2+1} &= \frac{b}{b^2+1}ab + \frac{c}{c^2+1}bc + \frac{a}{a^2+1}ca \\ \le \frac{ab+bc+ca}{2} &\le \frac{(a+b+c)^2}{6} \end{aligned}$$

and it remains to use that $a+b+c=3$.

3. Let $f(x,y)$ be the number written at the lattice point (x,y). Then

$$f(x,y) = \frac{f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)}{4}.$$

Assume that not all the numbers are equal. Then there are two points at distance 1 apart such that the numbers written there are different. Rotating the plane, if necessary, we may assume that $f(x_0+1,y_0) > f(x_0,y_0)$ for some $x_0, y_0 \in \mathbb{Z}$. Set

$$g(x,y) = f(x+1,y) - f(x,y).$$

Then $M = \sup_{x,y\in\mathbb{Z}} g(x,y) \in (0,1]$ and

$$g(x,y) = \frac{g(x+1,y)+g(x-1,y)+g(x,y+1)+g(x,y-1)}{4}.$$

In particular, if $g(a,b) \geq M - \varepsilon$, where $\varepsilon > 0$, then

$$\begin{aligned} g(a+1,b) &= 4g(a,b) - g(a-1,b) - g(a,b+1) - g(a,b-1) \\ &\geq 4(M-\varepsilon) - 3M = M - 4\varepsilon \end{aligned}$$

and we get by induction that $g(a+n,b) \geq M - 4^n\varepsilon$ for any $n \in \mathbb{N}$.

Choose now $n \geq \dfrac{2}{M}$, $\varepsilon \in \left(0, \dfrac{M}{2.4^{n-1}}\right]$ and integers a and b such that $g(a,b) \geq M - \varepsilon$. Then

$$1 > f(a+n,b) > f(a+n,b) - f(a,b) = \sum_{k=0}^{n-1} g(a+k,b) \geq n\frac{M}{2} \geq 1,$$

which is a contradiction.

Remark. One can show that the statement of the problem remains true if the numbers at the lattice points are uniformly bounded from above or below.

4. We have $P_1(x) = P_2(x) = 1$, $P_3(x) = x+1$, $P_4(x) = x^2+1$ and $P_6(x) = x^4+1$. Hence the integers $n = 1,2,3,4,6$ have the desired property. We shall prove that these are the only solutions of the problem.

It is enough to show that for $n \geq 3$ the polynomial $P_n(x)$ has a divisor of the form $1+x^r$, $r \geq 1$, which is proper divisor for $n = 5$ and $n \geq 7$.

If $n \geq 3$ is a prime, this follows by the decomposition

$$P_n(x) = (1+x)(1+x^2+x^4+\cdots+x^{n-3}).$$

Note also that $P_4(x) = x^2+1$.

Now we shall use induction on n. Suppose that the statement is true for all $m \geq 3$ that are less than n. If $n \geq 6$ is a composite integer, then $n = mp$, where p is prime and $m \geq 3$. Two cases are possible.

Case 1. p divides m. Then we have that $A_n = \cup_{i=0}^{p-1}(A_m + im)$ and hence $P_n(x) = P_m(x)\sum_{i=0}^{p-1} x^{im}$. It remains to use that $P_m(x)$ has a divisor of the form $1+x^r$.

Case 2. p does not divide m. Using that

$$A_n = \left(\cup_{i=0}^{p-1}(A_m + im)\right) \setminus (pA_m)$$

and $x^{kp-1} = x^{p-1}(x^p)^{k-1}$, it follows that

$$P_n(x) = P_m(x)\sum_{i=0}^{p-1} x^{im} - x^{p-1}P_m(x^p).$$

By the induction assumption, $P_m(x)$ has a divisor of the form $1+x^r$. Therefore $1+x^{pr}$ divides $P_m(x^p)$. We shall consider two subcases.

a) If $p \geq 3$, then p is odd and $1+x^r$ divides $1+x^{pr}$ Hence $1+x^r$ divides $P_n(x)$.

b) If $p = 2$, we may assume that m is prime (otherwise, m has an odd prime divisor and we may go either to a) or to the first case). Then

$$P_n(x) = (1+x^{m+1})(1+x^2+x^4+\cdots+x^{m-3}).$$

It remains to observe that $P_n(x) = 1+x^r$ only for $n = 3, 4, 6$ which completes the solution.

Team selection test for 44. IMO

1. It is not difficult to see that the rectangles must be placed one over another so that any of them has two vertices on the sides AC and BC, and the first one has a base on the side AB.

Now we shall prove by induction that the sum of the areas of n such rectangles is maximal when the side AC is divided into $n+1$ equal parts by the vertices of the rectangles lying on it. Then the sum equals $\frac{n}{n+1}S$, where $S = S_{ABC}$.

Let $MNPQ$ be a rectangle with $M, N \in AB$, $P \in BC$, $Q \in AC$. Setting $\frac{CQ}{AC} = x$, it easily follows that

$$S_{MNPQ} = 2x(1-x)S.$$

Hence S_{MNPQ} is maximal if $x = \frac{1}{2}$, i.e., when Q is the midpoint of the side AC. This proves our statement for $n = 1$.

Assume that the statement holds true for some k and consider $k+1$ rectangles $M_iN_iP_iQ_i$ with $M_i, N_i \in P_{i-1}Q_{i-1}$ ($P_0 \equiv A$, $Q_0 \equiv B$) and $P_i \in BC$ и $Q_i \in AC, i = 1, 2, \ldots, k+1$. Setting $\frac{CQ_1}{AC} = x$, we get $S_{M_1N_1P_1Q_1} = 2x(1-x)S$. The induction assumption implies that $\sum_{i=2}^{k+1} S_{M_iN_iP_iQ_i}$ is maximal if $Q_1Q_2 = Q_2Q_3 = \cdots = Q_kQ_{k+1} = Q_{k+1}C$. Therefore

$$\sum_{i=2}^{k+1} S_{M_iN_iP_iQ_i} \leq \frac{kS_{Q_1P_1C}}{k+1} = \frac{kx^2S}{k+1}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} S_{M_iN_iP_iQ_i} &\leq \left(2x(1-x) + \frac{kx^2}{k+1}\right)S \\
&= \left[\frac{k+1}{k+2} - \frac{k+2}{k+1}\left(x - \frac{k+1}{k+2}\right)^2\right]S \\
&\leq \frac{k+1}{k+2}S.
\end{aligned}$$

The equality is attained if $x = \frac{k+1}{k+2}$, i.e., when the points $Q_1, Q_2, \ldots, Q_{k+1}$ divide the side AC into equal parts.

2. It follows by

$$f(x^2 + y + f(y)) = 2y + (f(x))^2 \tag{1}$$

that the function f is surjective. Note also that $(f(x))^2 = (f(-x))^2$. In particular, we may choose a such that $f(a) = f(-a) = 0$. Setting $x = 0$, $y = \pm a$ in (1) gives $0 = f(\pm a) = (f(0))^2 \pm 2a$, i.e., $a = 0$. Plugging $y = -\dfrac{(f(x))^2}{2}$ in (1), we get that $f(x^2 + y + f(y)) = 0$ and therefore $y + f(y) = -x^2$. Thus the function $y + f(y)$ takes any non-positive value. Since $f(0) = 0$, it follows from (1) that

$$f(x^2) = ((f(x))^2 \geq 0 \quad \text{and} \quad f(y + f(y)) = 2y.$$

Setting $z = x^2$, $t = y + f(y)$, and using again (1) we deduce that $f(z + t) = f(z) + f(t)$ for any $z \geq 0 \geq t$. For $z = -t$ we get $f(-t) = -f(t)$ and then it is easy to check that $f(z + t) = f(z) + f(t)$ for any z and t. Since $f(t) \geq 0$ for $t \geq 0$, it follows that f is an increasing function. Suppose that $f(y) > y$ for some y. Then $f(f(y)) \geq f(y)$ and we get

$$2y = f(y + f(y)) = f(y) + f(f(y)) > 2f(y),$$

a contradiction. Hence $f(y) \leq y$. We see in the same way that $f(y) \geq y$, so $f(x) \equiv x$. This function obviously satisfies (1).

3. Let $A_1A_2\ldots A_n$ be a convex n-gon. Denote by $\mathcal{B}$ the set of the connected vertices and let $B_1B_2\ldots B_k$ be its convex hull. We shall prove by induction on n that there is a map with the desired properties that in addition sends two given adjacent vertices of the n-gon to two given adjacent vertexes of $B_1B_2\ldots B_k$.

The base of the induction $n = 3$ is obvious. Suppose that our statement is true for any $k < n$. To prove it for n, it is enough to find a map for n that sends A_1 and A_2 to B_1 and B_2, respectively. Note that there is a unique point A_i that is connected with A_1 and A_2 (otherwise, some segments will have a common interior point). Consider the points $X_1, X_2, \ldots, X_s$ from $\mathcal{B}$ such that any of the triangles $B_1X_iB_2$ contains no points of $\mathcal{B}$. It is easy to find a point among them, say X_l, such that the interiors of $\measuredangle B_1B_2X_l$ and $\measuredangle B_2B_1X_l$ contain at most $n - i$ and $i - 3$ points of $\mathcal{B}$, respectively. It is clear now that there are a line through X_l and an interior point of the segment B_1B_2 that divides the set $\mathcal{B}$ into two subsets $\mathcal{B}_1$ and $\mathcal{B}_2$, containing $n - i + 1$ and $i - 2$ points, respectively. Let B_1X_l and B_2X_l be sides of the convex hulls of these two sets. If A_i is the corresponding point to X_l, then applying the induction assumption to the sets $A_2A_3\ldots A_i$ and $\mathcal{B}_2 \cup \{X_l\}$, and to $A_iA_{i+1}\ldots A_nA_1$ and $\mathcal{B}_1 \cup \{X_l\}$, we see that the statement is true for n points. This completes the solution of the problem.

4. The answer is *no*. We shall prove by induction that for any $k \geq 3$ there is a permutation $a_1, a_2, \ldots, a_k$ of $1, 2, \ldots, k$ such that

(1) $\qquad a_m + a_n \neq 2a_{\frac{m+n}{2}}$ for any $1 \leq m < n \leq k$ of the same parity.

For $k = 3$ and $k = 4$ take the permutations $1, 3, 2$ and $1, 3, 2, 4$, respectively. Assume that our statement is true for any integer less than k. Start with the

following permutation of $1, 2, \dots, k$: the odd numbers are in the first block, the even numbers that are not divisible by 4 are in the second block, etc. For example, if $k = 12$, then we have $1, 3, 5, 7, 9, 11$; $2, 6, 10$; $4, 12$; 8.

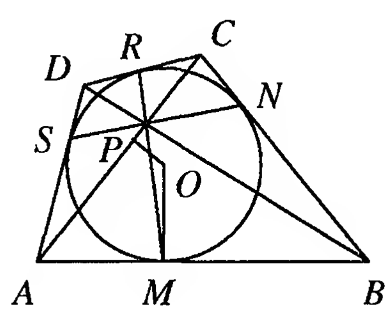
If a_m and a_n are in different blocks, set $a_m = 2^s b$ and $a_n = 2^t c$, where $t > s > 0$ and b, c are odd integers. Then $\frac{a_m + a_n}{2} = 2^{s-1}(b + 2^{t-s}c)$ is in the block before that of a_m, whence $a_{\frac{m+n}{2}}$ is in a block between these of a_m and a_n. So (1) holds.

It remains to reorder the integers in any block in such a way that (1) is satisfied for the numbers in this block. Consider the $(r+1)$-th block: $2^r, 3.2^r, \dots, (2d-1)2^r$, where $2d - 1 \leq k$. By the induction assumption there is a permutation $b_1, b_2, \dots, b_d$ of $1, 2, \dots, d$ satisfying (1). Set $c_i = 2b_i - 1$ for $i = 1, 2, \dots, d$ and consider the permutation $2^r c_1, 2^r c_2, \dots, 2^r c_{2d-1}$. Then

$$\frac{c_m + c_n}{2} = b_m + b_n - 1 \neq 2b_{\frac{m+n}{2}} - 1 = c_{\frac{m+n}{2}},$$

which completes the proof.

5. We shall use the following fact: if α, β, γ and δ are angles such that $\sin\alpha \sin\delta = \sin\beta \sin\gamma$ and $\alpha + \beta = \gamma + \delta < 180°$, then $\alpha = \gamma$ and $\beta = \delta$.



Denote by M, N, R and S the tangent points of the incircle of $ABCD$ centered at O with the sides AB, BC, CD and DA, respectively. Then the points A, M, O, P and S lie on the circle with diameter AO and $\sphericalangle APM = \frac{\overset{\frown}{AM}}{2} = \frac{\overset{\frown}{AS}}{2} = \sphericalangle APS$.

Analogously, $\sphericalangle CPR = \sphericalangle CPN$ and hence $\sphericalangle SPR = \sphericalangle MPN$ The Sine theorem for $\triangle BPM$ and $\triangle BPN$ gives

$$\frac{\sin \sphericalangle MPB}{\sin \sphericalangle PMB} = \frac{BM}{BP} = \frac{BN}{BP} = \frac{\sin \sphericalangle BPN}{\sin \sphericalangle BNP}$$

and therefore $\frac{\sin \sphericalangle MPB}{\sin \sphericalangle NPB} = \frac{\sin \sphericalangle PMB}{\sin \sphericalangle PNB}$. Since

$$\begin{aligned} \sphericalangle PMB &= 180° - \sphericalangle AMP = 180° - \sphericalangle AOP, \\ \sphericalangle PNB &= 180° - \sphericalangle CNP = 180° - \sphericalangle COP, \end{aligned}$$

then $\frac{\sin \sphericalangle MPB}{\sin \sphericalangle NPB} = \frac{\sin \sphericalangle AOP}{\sin \sphericalangle COP}$. We get in the same way that

$$\frac{\sin \sphericalangle SPD}{\sin \sphericalangle RPD} = \frac{\sin \sphericalangle AOP}{\sin \sphericalangle COP}.$$

Applying the fact mentioned above with $\alpha = \sphericalangle MPB, \beta = \sphericalangle NPB, \gamma = \sphericalangle SPD$ and $\delta = \sphericalangle RPD$ we conclude that $\sphericalangle MPB = \sphericalangle SPD$ and therefore $\sphericalangle APB = \sphericalangle APM + \sphericalangle MPB = \sphericalangle APS + \sphericalangle SPD = \sphericalangle APD$.

6. We shall use that if $\left(\sqrt{a^2-1}+1\right)^k = x_k\sqrt{a^2-1}+y_k$, then all the solutions of Pell's equation $(a^2-1)x^2+1=y^2$ are (x_k, y_k). This implies that $(a^2-1)x^2+1$ is a perfect square if and only if x is a term of the sequence defined by $x_0=0$, $x_1=1$ and $x_{k+2}=2ax_{k+1}-x_k$ for $k\geq 0$.

Since $m(m+1)(m+2)(m+3)+1=(m^2+3m+1)^2$, then $n(n+1)^2(n+2)^3(n+3)^4+1 = [(n+1)^2-1][(n+1)(n+2)(n+3)^2]^2+1$ is a perfect square. Applying the property mentioned above with $a=n+1$ gives that $(n+1)(n+2)(n+3)^2$ is a term of the sequence defined by $x_0=0$, $x_1=1$ and $x_{k+2}=(2n+2)x_{k+1}-x_k$ for $k\geq 0$. Now it is easy to see by 'induction on k that the remainders of any x_k modulo $2n+1$ and $2n+3$ are 0, 1 or -1. Hence $(n+1)(n+2)(n+3)^2\equiv 0,\pm 1 \pmod{2n+1}$ and then

$$(2n+2)(2n+4)(2n+6)^2\equiv 0,\pm 16 \pmod{2n+1}.$$

Using that $2n+2\equiv 1 \pmod{2n+1}$, $2n+4\equiv 3 \pmod{2n+1}$ and $2n+6\equiv 5 \pmod{2n+1}$, it follows that $2n+1$ divides 75, 59 or 91. Repeating the same arguments, we get that $2n+3$ divides 7, 9 or 25. The only numbers satisfying both conditions are $n=1,2,3$. Now direct verifications complete the solution.

Bulgarian Mathematical Competitions 2004

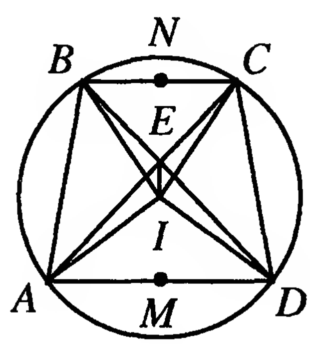
Winter Mathematical Competition

9.1. The equation has two distinct positive roots if and only if

$$\left|\begin{array}{l} D = a^3 - a^2 - 9a + 9 > 0 \\ x_1 + x_2 = \dfrac{6}{a^2 - a - 9} > 0 \\ x_1 x_2 = \dfrac{-a}{a^2 - a - 9} > 0 \end{array}\right. .$$

The first inequality is satisfied for $a \in (-3, 1) \cup (3, +\infty)$, the second one – for $a \in ((1 - \sqrt{37})/2, (1 + \sqrt{37})/2)$, and the third – for every $a \in (-\infty, 0)$. Therefore the required values of a are $a \in (-3, (1 - \sqrt{37})/2)$.

9.2. Denote the midpoints of AD and BC by M and N, respectively. Without loss of generality we may assume that I lies in the interior of $AMNB$ and E lies in the interior to $MDCN$. Then we have



$$\begin{aligned} S_{MIN} &= S_{AMNB} - S_{ABI} - S_{AMI} - S_{BNI} \\ &= S_{AMNB} - S_{ABI} - \frac{1}{2}(S_{ADI} + S_{BCI}) \\ &= S_{AMNB} - \frac{1}{2}S_{ABI} - \frac{1}{2}(S_{ABCD} - S_{CDI}). \end{aligned}$$

Analogously, $S_{MEN} = S_{MDNC} - \frac{1}{2}S_{DCE} - \frac{1}{2}(S_{ABCD} - S_{ABE})$. Now using $S_{MIN} = S_{MEN}$ we get

$$S_{AMNB} - \frac{1}{2}S_{ABI} - \frac{1}{2}(S_{ABCD} - S_{CDI}) = S_{MDNC} - \frac{1}{2}S_{DCE} - \frac{1}{2}(S_{ABCD} - S_{ABE}),$$

$$\begin{aligned} &\frac{1}{2}S_{ADN} + \frac{1}{2}S_{ABC} - \frac{1}{2}S_{ABI} - \frac{1}{2}(S_{ABCD} + \frac{1}{2}S_{CDI}) \\ &= \frac{1}{2}S_{ADN} + \frac{1}{2}S_{CBD} - \frac{1}{2}S_{DCE} - \frac{1}{2}(S_{ABCD} + \frac{1}{2}S_{ABE}), \\ &S_{ABC} - S_{CBD} + S_{DCE} - S_{ABE} = S_{ABI} - S_{CDI}, \\ &S_{ABE} - S_{CDE} + S_{DCE} - S_{ABE} = S_{ABI} - S_{CDI}. \end{aligned}$$

Therefore $S_{ABI} = S_{CDI}$, whence $AB = CD$.

9.3. Denote by $f(n)$ the least number of colors such that the integers $1, 2, \ldots, n$ can be colored in the required way. We shall prove that $f(n) = \lfloor (k+1)/2 \rfloor$, where $2^{k-1} \le n < 2^k$. Observe that in the sequence $1, 2, 2^2, \ldots, 2^{k-1}$ we have no three numbers of the same color. This means that $f(n) \ge \lfloor (k+1)/2 \rfloor$.

Consider the following coloring by $\lfloor (k+1)/2 \rfloor$ colors (each color is identified with an integer among $1, 2, \ldots, \lfloor (k+1)/2 \rfloor$. If $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t} \le n$, where p_i are primes, then we have $h(m) := \alpha_1 + \cdots + \alpha_t < k$ and we can correctly color m by the color $\lfloor (h(m)+1)/2 \rfloor$. If a divides b and b divides c, then we have $h(a) < h(b) < h(c)$, i.e., $h(c) - h(a) \ge 2$. This means that the numbers a and c have different colors. Hence $f(n) = \lfloor (k+1)/2 \rfloor$. Now applying the above formula for $n = 2004$ we get $f(2004) = 6$.

10.1. Since $y_1 + y_2 = 1$ and $y_1 y_2 = 6a$, we have

$$(x^2 - y_1 x + a)(x^2 - y_2 x + a)$$

$$= x^4 - (y_1 + y_2)x^3 + (2a + y_1 y_2)x^2 - a(y_1 + y_2)x + a^2 = f(x).$$

b) The roots of $f(x) = 0$ are the roots x_1, x_2 and x_3, x_4 of the equations

$$x^2 - y_1 x + a = 0 \quad \text{and} \quad x^2 - y_2 x + a = 0,$$

respectively. It is easy to see that x_1, x_2, x_3 and x_4 are real, distinct and positive exactly when the following three conditions are simultaneously satisfied:

1) $a \neq 0$, $y_1 \neq y_2$ are real $\iff a \neq 0$, $D = 1 - 24a > 0 \iff a \neq 0$, $a < \dfrac{1}{24}$;

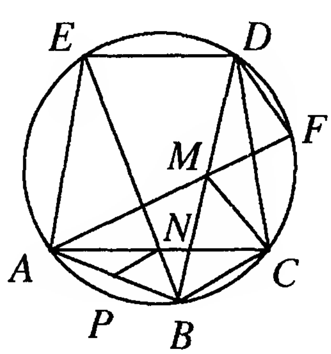
2) $D_1 = y_1^2 - 4a > 0$ and $D_2 = y_2^2 - 4a > 0 \iff y_1 > 10a$ and $y_2 > 10a$;

3) $x_1 + x_2 = y_1 > 0$, $x_1 x_2 = a > 0$, $x_3 + x_4 = y_2 > 0$ and $x_3 x_4 = a > 0$ $\iff y_1 > 0$, $y_2 > 0$ and $a > 0$.

These conditions are equivalent to $0 < a < \dfrac{1}{24}$, $y_1 > 10a$ and $y_2 > 10a$, which is the same as $0 < a < \dfrac{1}{24}$, $g(10a) > 0$ and $10a < \dfrac{1}{2} \iff 0 < a < \dfrac{1}{24}$, $4a(25a - 1) > 0$ and $a < \dfrac{1}{20}$. Hence the required values of a are $a \in \left(\dfrac{1}{25}, \dfrac{1}{24} \right)$.

10.2. Let BE meet AC at point N and P be the midpoint of AB. Set $\sphericalangle BAC = \sphericalangle BDC = \alpha$, $\sphericalangle ABE = \sphericalangle CBD = \beta$ and $\sphericalangle ADB = \sphericalangle ACB = \gamma$. Then $\triangle ABN \sim \triangle DBC$, and we conclude that $\triangle BPN \sim \triangle BMC$.

Let $\sphericalangle AMB = \sphericalangle BMC = \varphi$. We have from the above that $\sphericalangle BPN = \varphi$.



We shall use the following fact: if two chords of a circle bisect a third one and determine equal angles with it, then they are equal and their intersection point divides them into respectively equal parts (use congruent triangles or symmetry through a line). Let the ray $AM^{\rightarrow}$ meet the circle at point F. Then $CM = FM$ and therefore $\triangle BMC \cong \triangle DMF$. Hence we have $BC = DF$ and $\sphericalangle MAD = \sphericalangle BDC = \alpha$.

It follows from ΔAMD that $\varphi = \alpha + \gamma$ and using ΔAPN we get $\sphericalangle ANP = \varphi - \alpha = \gamma = \sphericalangle ACB$. Hence $NP || BC$, which means that N is the midpoint of AC. This completes the proof.

10.3. Suppose that n has the required property. For every $j = 1, 2, \ldots, n$ denote by q_j the least prime divisor of a_j and let $q = \max\limits_{1 \le i \le n} q_i$. Without loss of generality we may assume that $q = q_1$. Then

$$(3n+1)^2 \ge a_1 \ge q_1^2 \ge p_n^2,$$

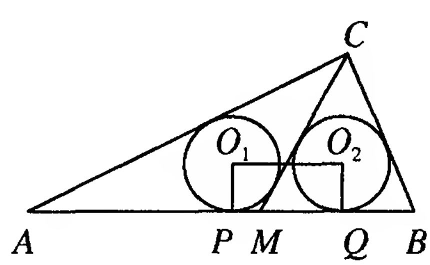
where p_n is the n-th prime number. Therefore we have $p_n \le 3n+1$. It is easy to show (by induction) that $p_n > 3n+1$ for every $n \ge 15$. Hence $n \le 14$. Since the set $\{2^2, 3^2, 5^2, \ldots, p_{14}^2\}$ has the required properties, we conclude that $n = 14$.

11.1. Setting $y = 2^x$ we have to find all values of a such that the equation

$$y^2 - (a^2+3a-2)y + 3a^3 - 2a^2 = 0 \iff (y-a^2)(y-3a+2) = 0$$

has exactly one positive root. Obviously $a = 0$ is not a solution. For $a \neq 0$ the equation has a positive root $y_1 = a^2$ It is unique if either $y_2 = 3a - 2 \le 0$ or $y_2 = y_1$. In the first case we obtain $a \le \dfrac{2}{3}$ and in the second one we have $a^2 = 3a - 2$, whence $a_1 = 1$ and $a_2 = 2$.

Finally, $a \in (-\infty; 0) \cup \left(0; \dfrac{2}{3}\right) \cup \{1\} \cup \{2\}$.



11.2. Set $AB = c, BC = a, CA = b, 2p = a + b + c$ and let the inradii of ΔAMC and ΔBMC be equal to r.

a) Since $S_{ABC} = r\left(\dfrac{a+b+c}{2} + CM\right)$ and

$S_{PQO_2O_1} = r\,(MQ + MP) =$

$$= r\left(\frac{MB + CM - a}{2} + \frac{MA + CM - b}{2}\right) = r\left(CM + \frac{c}{2} - \frac{a+b}{2}\right),$$

we get $\dfrac{a+b+c}{2} + CM = 6\left(CM + \dfrac{c}{2} - \dfrac{a+b}{2}\right)$. This is equivalent to the required equality.

b) First we shall prove that $CM = \sqrt{p(p-c)}$. If $\sphericalangle AMC = \varphi$, then

$$PM + QM = r(\cot\frac{\varphi}{2} + \tan\frac{\varphi}{2}) = \frac{2r}{\sin\varphi}$$

and hence

$$2r = PQ\sin\varphi = \left(CM + \frac{c}{2} - \frac{a+b}{2}\right)\sin\varphi = (CM - (p-c))\sin\varphi.$$

We have also that

$$S_{ABC} = r(p + CM) = \frac{(CM - (p-c))\sin\varphi}{2}(p + CM).$$

Since $S_{ABC} = \dfrac{c.CM\sin\varphi}{2}$, we get

$$c.CM = (CM - (p-c))(p + CM)$$

which is equivalent to $CM = \sqrt{p(p-c)}$.

Therefore

$$10CM + 5c = 7(a+b) \quad \text{and} \quad 4CM^2 = (a+b+c)(a+b-c).$$

Setting $\dfrac{CM}{c} = m$ and $\dfrac{a+b}{c} = n$ gives

$$10m + 5 = 7n \quad \text{and} \quad 4m^2 = (n+1)(n-1) = n^2 - 1.$$

This system has solutions $m = \dfrac{3}{8}$, $n = \dfrac{5}{4}$ and $m = \dfrac{2}{3}$, $n = \dfrac{5}{3}$. Therefore the required ratio is equal to $\dfrac{5}{4}$ or $\dfrac{5}{3}$.

11.3. We shall prove by induction on m that $a_{n+m} = a_m a_{n+1} - a_{m-1}a_n$ for every two positive integers $n, m \geq 2$. For $m = 2$ this is the given recurrence relation. If for some $m \geq 2$ the equality is satisfied for every n, then

$$\begin{aligned} a_{m+1+n} &= a_{m+(n+1)} = a_m a_{n+2} - a_{m-1}a_{n+1} = \\ &= a_m(a a_{n+1} - a_n) - a_{m-1}a_{n+1} = \\ &= (a a_m - a_{m-1})a_{n+1} - a_m a_n = a_{m+1}a_{n+1} - a_m a_n, \end{aligned}$$

which completes the induction.

The recurrence relation shows that $\gcd(a_n, a_{n-1}) = 1$ for every $n \geq 2$. This and $a_{n+m} = a_m a_{n+1} - a_{m-1}a_n$ imply that $\gcd(a_{m+n}, a_m) = \gcd(a_m, a_n)$. Using induction again we conclude that for every two positive integers m and n we have $\gcd(a_m, a_n) = a_{\gcd(m,n)}$. Now the assertion follows immediately: if $1 < n_1 < n_2 < \cdots < n_k < \cdots$ is an infinite sequence of relatively prime integers then $\gcd(a_{n_i}, a_{n_j}) = a_{\gcd(n_i,n_j)} = a_1 = 1$, i.e., $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$ are relatively prime. Therefore the set of their prime factors is infinite.

12.1. a) We have $a_2 = a_1 + \dfrac{1}{a_1} \geq 2$. If $a_n \geq n$, then

$$a_{n+1} - n - 1 = a_n + \frac{n}{a_n} - n - 1 = \frac{(a_n - 1)(a_n - n)}{a_n} \geq 0$$

and the assertion follows by induction.

b) Let $n \geq 2$. It follows from a) that $a_{n+1} \leq a_n + 1$. Then $a_n \leq a_2 + n - 2$, whence $1 \leq \frac{a_n}{n} \leq 1 + \frac{a_2 - 2}{n}$. Therefore the sequence $\left(\frac{a_n}{n}\right)_{n\geq 1}$ is convergent and its limit equals 1.

Remark. One can prove the stronger statement that $\lim_{n\to\infty} (a_n - n) = 0$.

12.2. a) If $\triangle ABC$ is acute, then by the Law of cosines for $\triangle AHB$ we get that

$$AB^2 = AH^2 + BH^2 - 2AH.BH\cos(\pi - \gamma).$$

Since $AB = 2R\sin\gamma$ and $CH = 2R\cos\gamma$ (by the Extended Law of sines), we obtain $AB^2 + CH^2 = 4R^2$. Therefore

$$4R^2 = AH^2 + BH^2 + CH^2 + \frac{AH.BH.CH}{R}.$$

Then $4R^3 = 7R + 3$, i.e. $(R+1)(2R+1)(2R-3) = 0$, whence $R = \frac{3}{2}$.

If $\triangle ABC$ is obtuse, then we get analogously that

$$4R^2 = AH^2 + BH^2 + CH^2 - \frac{AH.BH.CH}{R}$$

and therefore $4R^3 = 7R - 3$, i.e. $(R-1)(2R-1)(2R+3) = 0$. Since $3 = AH.BH.CH < (2R)^3$, we conclude that $R = 1$.

The existence of $\triangle ABC$ with $R = \frac{3}{2}$ and $R = 1$ follows from b).

b) Denote by S the area of $\triangle ABC$. Since $S = \frac{AB.BC.CA}{4R}$, we have

$$S^2 = \frac{(4R^2 - AH^2)(4R^2 - BH^2)(4R^2 - CH^2)}{16R^2}.$$

Setting $x = AH^2, y = BH^2, z = CH^2$ and $t = 4R^2$, we get

$$S^2 = \frac{t^3 - 7t^2 + t(xy + yz + zx) - 9}{4t}.$$

Without loss of generality we may assume that $x \geq y \geq z$. Then $x \geq \frac{7}{3}$ and therefore

$$xy + yz + zx = \frac{9}{x} + x(7 - x) = 15 - \frac{(x-3)^2(x-1)}{x} \leq 15,$$

where the equality is attained if $x = 3$. Hence

$$S^2 \leq \frac{t^3 - 7t^2 + 15t - 9}{4t}.$$

Since $R = \frac{3}{2}$ or $R = 1$, we conclude that $S_{max} = \sqrt{8}$ and it is achieved for an acute $\triangle ABC$ with $R = \frac{3}{2}$, $AH = BH = \sqrt{3}$ and $CH = 1$. The sides of this triangle are $\sqrt{6}$, $\sqrt{6}$ and $\sqrt{8}$.

12.3. First we shall prove the following:

LEMMA. *Let $p \geq 3$ be an odd divisor of b. Then there exists an odd prime q that divides $(b+1)^p - 1$ but does not divide b.*

Proof of the lemma. If $b = pc$, then

$$\begin{aligned}(b+1)^p - 1 &= b((b+1)^{p-1} + \cdots + b + 1)\\ &= b(Bb^2 + \frac{p(p-1)}{2}b + p) = bp(b(Bc + \frac{p-1}{2}) + 1)) = bpd\end{aligned}$$

and it remains to choose a prime divisor of d. Note that d is odd (if b is even, then $d = bK + 1$ is odd; if b is odd, then $(b+1)^p - 1$ is odd, and hence d is odd, too).

We shall prove now that if $a \neq 2^k + 1$, then there exists a sequence of odd primes $p_1, p_2, \ldots$ such that p_1 divides $a - 1$, and if $P_n = a^{p_0 p_1 \ldots p_n} - 1$ (here $p_0 = 1$), then p_{n+1} divides P_n, but does not divide P_{n-1}, $n \geq 1$.

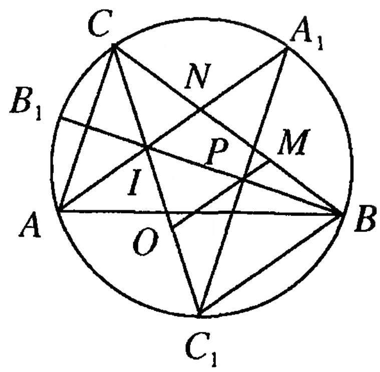
Let p_1 be an odd prime divisor of $a - 1$ and we have already chosen the primes $p_1, \ldots, p_k$. Applying the lemma for $b = P_k$ and $p = p_k$, we find an odd prime p_{k+1} that divides P_k but does not divide P_{k-1}.

Since P_{k-1} is divisible by $p_1, p_2, \ldots, p_k$, we conclude that p_{k+1} differs from them. Therefore the numbers $p_1, p_2, \ldots p_k$ have the required property.

If $a = 2^l + 1$, $l \geq 2$, then $a^2 \neq 2^m + 1$ and it remains to multiply by 2 the numbers already found for a^2.

Remark. It can be proved that if n divides $2^n - 1$, then $n = 1$, and if n divides $3^n - 1$ then $n = 1$, $n = 2$, or n is divisible by 4. The above solution shows that for $a = 3^4$ there exist infinitely many positive square-free odd integers n such that $4n$ divides $3^{4n} - 1$.

8.1. Set $\sphericalangle CAB = \alpha$, $\sphericalangle ABC = \beta$ and $\sphericalangle BCA = \gamma$. Then



$$\begin{aligned}\sphericalangle IPC_1 &= \frac{1}{2}(\overset{\frown}{BA_1} + \overset{\frown}{C_1B_1})\\ &= \frac{1}{2}(\overset{\frown}{BA_1} + \overset{\frown}{AC_1} + \overset{\frown}{AB_1})\\ &= \frac{1}{2}(\alpha+\beta+\gamma) = 90^\circ.\end{aligned}$$

Hence O is the midpoint of the segment IC_1 and it follows that

$$\sphericalangle IOP = 2 \sphericalangle IC_1P = \overset{\frown}{CA_1} = \alpha.$$

We also have $\sphericalangle CC_1B = \alpha$ and therefore $OP||C_1B$. Since $C_1O = OI$ and $BM = MN$, we conclude that $IN||C_1B$, i.e. $\sphericalangle CIA_1 = \alpha$. On the other hand,

$$\sphericalangle CIA_1 = \frac{1}{2}(\overset{\frown}{CA_1} + \overset{\frown}{AC_1}) = \frac{1}{2}(\alpha+\gamma) = 90^\circ - \frac{\beta}{2}.$$

This implies that $\alpha = 90^\circ - \dfrac{\beta}{2} = \dfrac{\alpha+\gamma}{2}$, i.e. $\alpha = \gamma$. Since $\alpha = 2\beta$, we obtain that $\alpha = \gamma = 72^\circ$ and $\beta = 36^\circ$.

8.2. Denote by x the number of African teams. Then the number of European teams equals $x+9$. The African teams played each other $\dfrac{(x-1)x}{2}$ games and therefore the points won by them are $\dfrac{(x-1)x}{2} + k$, where k is the number of wins over European teams.

Further, the points won by the Europeans are $\dfrac{(x+8)(x+9)}{2} + x(x+9) - k$. Thus,

$$9\Big(\frac{(x-1)x}{2} + k\Big) = \frac{(x+8)(x+9)}{2} + x(x+9) - k,$$

and so $3x^2 - 22x + 10k - 36 = 0$. Since x is a positive integer, we have that $121 - 3(10k-36) = 229 - 30k$ is a perfect square. Then $k \le 7$ and a direct verification shows that we obtain perfect squares only for $k = 2$ and $k = 6$. For $k = 2$ we have $x = 8$ and therefore the best African team could have at most $7 + 2 = 9$ points.

For $k = 6$ we get $x = 6$ and therefore there are 6 African and 15 European teams. In this case the best African team has at most $5+6 = 11$ points, which happens if it wins over all other African teams and 6 European teams (the

other African teams lost their games against all European teams). Finally, the answer is 11.

8.3. a) Yes. Here is an example:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

b) No. We have 11 possibilities for these sums: $0, \pm1, \pm2, \pm3, \pm4$ и ±5. Denote by a_i the sum of the numbers in the i-th row and by b_j – the sum of the numbers in the j-th column. Obviously

$$a_1 + a_2 + a_3 + a_4 + a_5 = b_1 + b_2 + b_3 + b_4 + b_5,$$

which shows that the number of odd sums a_i and the number of odd sums b_j are of the same parity. Therefore all odd sums must be achieved.

Without loss of generality we may assume that $b_1 = 5$ and then none of a_i equals -5. Thus, we may suppose that $b_2 = -5$. At least one of the sums equals 4 or -4 and suppose it is 4. This is possible only if there is a column with four 1's and one 0. Let $b_3 = 4$ and the zero is in the last row. Therefore $a_i \neq -3$ for every i and we may assume that $b_4 = -3$. Then in the fourth column we have at least three -1's. If they are in the first four rows we may assume that they are in the first three rows. So a_1, a_2, a_3 is a permutation of $-1, 0, 1$. Therefore $b_5 \neq 3$ and since $a_5 \neq 3$, it follows that $a_4 = 3$, i.e., there are 1's in the last two cells of the fourth row. Since $b_4 = -3$, the number in the last cell of the fourth column is -1. Now every possibility for the number in the 5-th row and 5-th column leads to a contradiction.

It remains to consider the case when there are at most two -1's in the first four cells of the fourth column. We may assume that the numbers in the fourth column are consequently -1, -1, 0, 0 and -1. Since the sum of the first four numbers of the rows equal 0, 0, 1, 1 and -1 we have that none of the rows equals 3 and therefore $b_5 = 3$. We must use different numbers for rows 1 and 2 and for rows 3 and 4, so we may have at most three 1's. Hence we may assume that the numbers in the fifth column are 1, 0, 1, 0, 1 and then $a_1 = a_4 = 1$, a contradiction.

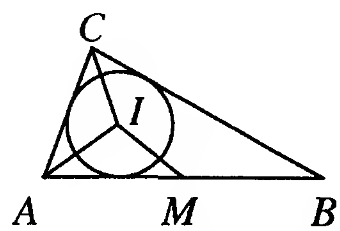
9.1. a) If $a = 0$, then $x = -y$ and hence $2x^2 = 2$. It follows that $(x, y) = (1, -1)$ or $(x, y) = (-1, 1)$.

b) We know from a) that $a = 0$ is one of the desired numbers. Let $a \neq 0$. Setting $x + y = p$, $xy = q$, we have $p = aq$ and $p^2 - 2q = a^2 + 2$. Then $a^2q^2 - 2q - a^2 - 2 = 0$ and hence $(p, q) = (-a, -1)$ or $(p, q) = \left(\frac{a^2+2}{a}, \frac{a^2+2}{a^2}\right)$. Note that these pairs of numbers are different. The first case $x + y = -a$, $xy = -1$ leads to the quadratic equation $z^2 + az - 1 = 0$ which has two distinct real roots z_1 and z_2. It follows that $(x, y) = (z_1, z_2)$ and $(x, y) = (z_2, z_1)$ are

solution of the given system. Thus we have to find all a for which the second case is impossible.This means that the discriminant of the quadratic equation $z^2 - \frac{a^2+2}{a}z + \frac{a^2+2}{a^2} = 0$ is negative, i.e., $a \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$. So the answer to b) is $a \in (-\sqrt{2}, \sqrt{2})$.

9.2. We may assume that $AC < BC$. Since $\sphericalangle ACI$ and $\sphericalangle AMI$ are acute, then $\triangle ACI \cong \triangle AMI$.

Hence $AC = AM$ and $\sphericalangle AIC = \sphericalangle AIM$, i.e.



$$\sphericalangle CIM = 360° - 2\sphericalangle AIC = 180° - \sphericalangle ABC.$$

Note that $\sphericalangle ABC$ is maximal if BC is tangent to the circle with center A and radius AM.

Then $\sphericalangle ACB = 90°$, $\sphericalangle ABC = 30°$, and hence the least possible value of $\sphericalangle CIM$ is $150°$.

9.3. Note that $k^{p-1} \equiv 0 (\text{mod } p)$ if p divides k and $k^{p-1} \equiv 1 (\text{mod } p)$ otherwise (by Fermat's little theorem). Then

$$0 \equiv 1^{p-1} + 2^{p-1} + \cdots + 2004^{p-1} \equiv 0.\left[\frac{2004}{p}\right] + 1.\left(2004 - \left[\frac{2004}{p}\right]\right) (\text{mod } p),$$

which implies

$$2004 \equiv \left[\frac{2004}{p}\right] (\text{mod } p) \qquad (1)$$

(in particular, $p < 2004$). Let $2004 = qp + r$, where $0 \le r \le p - 1$. Then $\left[\frac{2004}{p}\right] = \left[q + \frac{r}{p}\right] = q$ and (1) is equivalent to $r \equiv q (\text{mod } p)$.

For $q < p$, this congruence gives $r = q$. Then

$$2004 = (p+1)q \le p^2 - 1$$

and therefore $p \ge 47$. Since $p + 1$ divides $2004 = 3.4.167$, we get $p = 2003$ which is a solution of the problem.

For $q \ge p$, we have that $2004 \ge pq \ge p^2$, i.e., $p \le 43$. A direct verification of (1) shows that $p = 17$ is the only solution in this case.

10.1. a) Using Vieta's formulas we get $x_1^2 + x_1x_2 + x_2^2 = (x_1+x_2)^2 - x_1x_2 = 4$. Hence $|x_1^3 - x_2^3| \le 4 \iff |x_1 - x_2| \le 1$. Let $D = 16 - 3a^2$ be the the discriminant of $f(x)$. Then $D \ge 0$ and $|x_1 - x_2| = \sqrt{D}$.It follows that $0 \le 16 - 3a^2 \le 1$, and therefore $a \in \left[-\frac{4\sqrt{3}}{3}, -\sqrt{5}\right] \cup \left[\sqrt{5}, \frac{4\sqrt{3}}{3}\right]$.

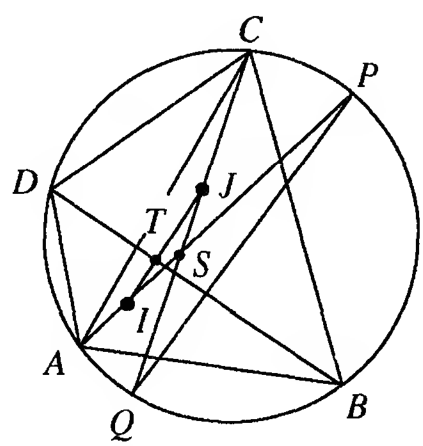
b) If $D = 16 - 3a^2 \le 0$, i.e. $a \in \left(-\infty, -\frac{4\sqrt{3}}{3}\right] \cup \left[\frac{4\sqrt{3}}{3}, \infty\right)$, then $f(x) \ge 0$ for every x. If $D > 0$, then $|x_1 - x_2| \le 1$ (since otherwise there is an

integer $x \in (x_1, x_2)$, i.e., $f(x) < 0$). Hence $a \in \left[-\frac{4\sqrt{3}}{3}, -\sqrt{5}\right] \cup \left[\sqrt{5}, \frac{4\sqrt{3}}{3}\right]$ and we have to find all a from these intervals such that $f(x) \geq 0$ for every integer x.

If $f(x) < 0$ for some integer x, then the distance between x and $\frac{a}{2}$ is at most $\frac{1}{2}$. Since $-\frac{3}{2} < -\frac{2\sqrt{3}}{3} \leq \frac{a}{2} \leq -\frac{\sqrt{5}}{2} < -1$ or $\frac{3}{2} > \frac{2\sqrt{3}}{3} \geq \frac{a}{2} \geq \frac{\sqrt{5}}{2} > 1$, we conclude that $x = \pm 1$ and by the inequalities $f(-1) \geq 0$ and $f(1) \geq 0$ we get $a \in \left[-\frac{4\sqrt{3}}{3}, \frac{-1-\sqrt{13}}{2}\right] \cup \left[\frac{1+\sqrt{13}}{2}, \frac{4\sqrt{3}}{3}\right]$. In conclusion, the desired values of a are

$$a \in \left(-\infty, \frac{-1-\sqrt{13}}{2}\right] \cup \left[\frac{1+\sqrt{13}}{2}, +\infty\right).$$

10.2. It is easy to see that if the points A, I, J and C are colinear, then $AB = AD$ and $BC = CD$. Hence $ABCD$ is a circumscribed quadrilateral.



Suppose that the points A, I, J and C are concyclic. Since $\sphericalangle AIC > \sphericalangle AIB$ or $\sphericalangle AIC > \sphericalangle AID$, it follows that $\sphericalangle AIC > 90°$. Analogously $\sphericalangle AJC > 90°$ and therefore $\sphericalangle AIC + \sphericalangle AJC > 180°$. It follows that the points I and J are on the same side of the line AC.

Let $S = AI \cap CJ$ and let the lines AI and CJ meet the circumcircle of $ABCD$ at points P and Q, respectively. Since P and Q are the midpoints of the arcs $\overset{\frown}{BCD}$ and $\overset{\frown}{BAD}$, respectively, it follows that $PQ \perp BD$.

On the other hand, $\sphericalangle SJI = \sphericalangle CAI = \sphericalangle CQP$, which implies that $IJ || PQ$. Therefore $IJ \perp BD$, which means that the incircles of $\triangle ABD$ and $\triangle BCD$ are tangent to each other at a point $T \in BD$. Then

$$DT = \frac{AD + BD - AB}{2} = \frac{BD + CD - BC}{2} \iff AB + CD = BC + AD,$$

i.e. $ABCD$ is a circumscribed quadrilateral.

Conversely, assume that $ABCD$ is a circumscribed quadrilateral. Note that if $I \in AC$, then $J \in AC$. Suppose that $I \notin AC$. It follows from the equality $AB + CD = BC + AD$ that the incircles of $\triangle ABD$ and $\triangle BCD$ are tangent to each other at a point of BD. Hence $IJ \perp BD \Rightarrow IJ || PQ \Rightarrow \sphericalangle SJI = \sphericalangle CQP = \sphericalangle CAI$, and therefore $ACJI$ is a cyclic quadrilateral.

10.3. See Problem 9.3.

11.1. The equation is defined if

$$4ax > 0, 4ax \neq 1, x - 3a > 0, x - 3a \neq 1. \qquad (*)$$

Setting $t = \log_{4ax}(x - 3a)$ we get the equation $t + \dfrac{1}{2t} = \dfrac{3}{2}$ with roots $t_1 = 1$ and $t_2 = \dfrac{1}{2}$.

If $\log_{4ax}(x - 3a) = 1$, then $x_1 = \dfrac{3a}{1-4a}$, $a \neq \dfrac{1}{4}$.

If $\log_{4ax}(x - 3a) = \dfrac{1}{2}$ we obtain the equation $x^2 - 6ax + 9a^2 = 4ax$ with roots $x_2 = 9a$ and $x_3 = a$.

Hence we have to find all a for which exactly two of the numbers $x_1 = \dfrac{3a}{1-4a}\left(a \neq \dfrac{1}{4}\right)$, $x_2 = 9a$ and $x_3 = a$ are different solutions of the given equation. It is clear that $a \neq 0$. We shall consider two cases.

1. Let $a > 0$. Then $x_3 - 3a = -2a < 0$, which implies that x_1 and x_2 have to be the solutions. Since $x_1 = \dfrac{3a}{1-4a}$ satisfies $(*)$, it follows that $4ax_1 = \dfrac{12a^2}{1-4a} > 0$, $4ax_1 = \dfrac{12a^2}{1-4a} \neq 1$, $x_1 - 3a = \dfrac{3a}{1-4a} - 3a > 0$ and $x_1 - 3a = \dfrac{3a}{1-4a} - 3a \neq 1$. This implies that $a < \dfrac{1}{4}$ and $a \neq \dfrac{1}{6}$. It is easy to check that if $a \neq \dfrac{1}{6}$, then $x_1 \neq x_2$ and x_2 satisfies $(*)$. So the desired numbers a in this case are

$$a \in \left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{4}\right).$$

2. Let $a < 0$. Then $x_2 - 3a = 6a < 0$, which implies that x_1 and x_3 have to be the two solutions. Since $x_1 = \dfrac{3a}{1-4a}$ again satisfies $(*)$, we find as in the first case that $a \neq -\dfrac{1}{2}$. Now it is easy to check that $x_1 \neq x_3$ and x_3 satisfies $(*)$. Thus

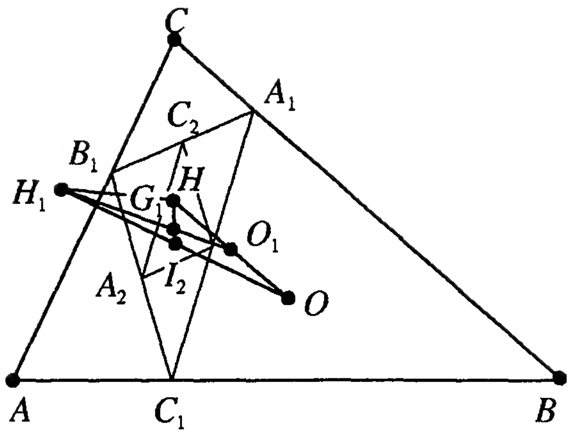
$$a \in \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right).$$

Combining both cases, it follows that the answer of the problem is

$$a \in \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{4}\right).$$

11.2. Denote by H the orthocenter of $\triangle ABC$, and by G_1 the centroid of $\triangle A_1B_1C_1$. Let O_1 be the midpoint of the segment OH. It is well-known that O_1 is the circumcenter of $\triangle A_1B_1C_1$, and H is its incenter. Then $\overrightarrow{H_1G_1} = 2\overrightarrow{G_1O_1}$.

Note that the dilation with center G_1 and ratio $-\dfrac{1}{2}$ maps $\triangle A_1B_1C_1$ into $\triangle A_2B_2C_2$ formed by the midpoints of the segment B_1C_1, A_1C_1 and A_1B_1. Hence the image of H under this dilation is the incenter I_2 of $\triangle A_2B_2C_2$. Since



G_1 is the centroid of $\triangle OHH_1$, it follows that I_2 is the midpoint of the segment OH_1.

11.3. Consider the more general problem for $n+1$ numbers instead of 100 and denote by $p_{k,n+1}$ the probability that a_m be equal to 1. If 1 is not the last number of a permutation of the positive integers $1, 2, \dots, n+1$, then $a_m = 1$ with probability $p_{k,n}$ (we assume that $p_{n,n} = 0$). Otherwise, $a_m = 1$ only if 2 and 3 are among the first k numbers of the permutation. Then

$$p_{k,n+1} = \frac{n}{n+1} p_{k,n} + \frac{1}{n+1} \cdot \frac{k}{n} \cdot \frac{k-1}{n-1}$$

and hence

$$(n+1)p_{k,n+1} = \sum_{j=k}^{n} ((j+1)p_{k,j+1} - jp_{k,j})$$

$$= k(k-1) \sum_{j=k}^{n} \frac{1}{j(j-1)} = \frac{k(n+1-k)}{n}.$$

It follows that $p_{k,n} = \dfrac{k(n-k)}{n(n-1)}$ and therefore $p_{k,100} = \dfrac{1}{4}$ for $k = 45$ and $k = 55$.

12.1. The equation of a common tangent line to the graphs of $f(x)$ and $g(x)$ at points $(x_1, f(x_1))$ and $(x_2, g(x_2))$ has the form

$$y = f(x_1) + f'(x_1)(x - x_1) = g(x_2) + g'(x_2)(x - x_2).$$

Hence $f'(x_1) = g'(x_2)$ and $f(x_1) - f'(x_1)x_1 = g(x_2) - g'(x_2)x_2$. Since $f'(x) = 2x - 2a$ and $g'(x) = -2x$, we get that $x_1 + x_2 = a$ and $x_1^2 + x_2^2 = 1$. Then $x_1 x_2 = \dfrac{a^2 - 1}{2}$ and hence x_1 and x_2 are the roots of the quadratic equation $x^2 - ax + \dfrac{a^2 - 1}{2} = 0$. Since the graphs of $f(x) = x^2 - 2ax$ and $g(x) = -x^2 - 1$ have two common tangent lines, it follows that they are disjoint, $x_1 \neq x_2$,

$a^2 < 2$, and the tangent points are $M(x_1, f(x_1)), N(x_2, f(x_2)), P(x_2, g(x_2))$ and $Q(x_1, g(x_1))$. Then

$$\begin{aligned} PQ^2 &= (x_1 - x_2)^2 + (g(x_1) - g(x_2))^2 \\ &= (x_1^2 + x_2^2 - 2x_1x_2)^2(1 + x_1 + x_2) = (2 - a^2)(1 + a^2), \end{aligned}$$

$$MQ = |f(x_1) - g(x_1)| = |x_1^2 - 2ax_1 + x_1^2 + 1| = 2 - a^2$$

and similarly

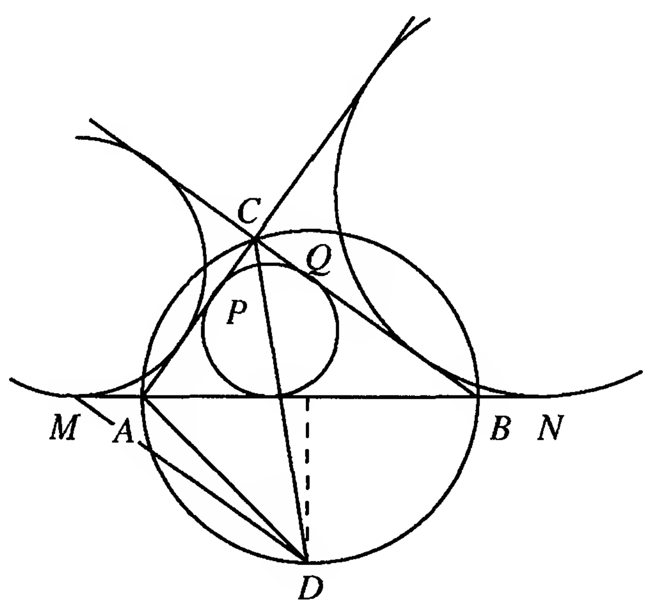
$$MN^2 = (2 - a^2)(1 + a^2), \ NP = 2 - a^2.$$

Hence $MNPQ$ is a parallelogram with perimeter

$$2(\sqrt{(2 - a^2)(1 + a^2)} + 2 - a^2) = 6.$$

Then $\sqrt{(2 - a^2)(1 + a^2)} = 1 + a^2$ and we get that $a = \pm\frac{\sqrt{2}}{2}$.

12.2. The perpendicular bisector of the segment AB and the bisector of $\measuredangle ACB$ meet at the midpoint D of the arc $\overset{\frown}{AB}$ of the circumcircle of $\triangle ABC$ which does not contain C.



Next we use the standard notation for the elements of $\triangle ABC$. Since $AM = BN = CP = CQ = p - c$, the condition of the problem is equivalent to the equality $DM = DP$. The Cosine theorem gives

$$DP^2 = DC^2 + CP^2 - 2DC.CP \cos\frac{\gamma}{2},$$

$$DM^2 = DA^2 + AM^2 + 2DA.AM \cos\frac{\gamma}{2}.$$

Subtracting these equalities, we get

$$DC - DA = 2(p - c)\cos\frac{\gamma}{2}. \tag{1}$$

On the other hand, we have $DC = \dfrac{(a+b)DA}{c}$ by the Ptolemy's theorem. Since $DA = \dfrac{c}{2\cos\frac{\gamma}{2}}$, we get $DC = \dfrac{a+b}{2\cos\frac{\gamma}{2}}$. Now (1) implies that $\cos^2\dfrac{\gamma}{2} = \dfrac{1}{2}$, i.e. $\gamma = 90°$.

Remark. The solution above shows that the points M, N, P and Q are concyclic if and only if $AC = BC$ or $\sphericalangle ACB = 90°$.

12.3. See Problem 11.3.

53. Bulgarian Mathematical Olympiad

Regional Round

9.1. Squaring the equation

$$\sqrt{(4a^2-4a-1)x^2-2ax+1}=1-ax-x^2 \tag{1}$$

gives the equation

$$x^2(x^2+2ax-3a^2+4a-1)=0$$

with roots $x_1=0$, $x_2=1-3a$ and $x_3=a-1$. It is clear that $x_1=0$ is a root of (1) for any a. On the other hand, $x_2=1-3a$ is a root of (1) if its right-hand side is non-negative, i.e., if

$$1-a(1-3a)-(1-3a)^2\geq 0 \iff 5a-6a^2\geq 0 \iff a\in\left[0,\frac{5}{6}\right].$$

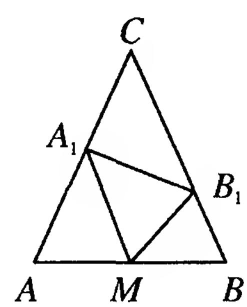
Analogously, $x_3=a$ is a root of (1) for $a\in\left[0,\frac{3}{2}\right]$. Two cases are possible.

Case 1. Some of the numbers x_1, x_2 and x_3 are equal. This implies that $a=\frac{1}{3},\frac{1}{2}$ or 1. It follows from above that $a=\frac{1}{3}$ and $a=\frac{1}{2}$ are solutions of the problem.

Case 2. The numbers x_1, x_2 and x_3 are pairwise different. Then it is easy to see that $a\in\left(\frac{5}{6},\frac{3}{2}\right]\setminus\{1\}$.

So, the desired values of a are $a=\frac{1}{3}$, $a=\frac{1}{2}$ and $a\in\left(\frac{5}{6},\frac{3}{2}\right]\setminus\{1\}$.

9.2. Denote by M the midpoint of AB. It follows that $\frac{AM}{BB_1}=\frac{AA_1}{BM}$. Then $\triangle AMA_1\sim\triangle BB_1M$ and hence $\frac{AA_1}{BM}=\frac{MA_1}{B_1M}$, i.e., $\frac{AA_1}{AM}=\frac{MA_1}{MB_1}$. Moreover, $\sphericalangle AA_1M=\sphericalangle BMB_1$ and therefore



$$\begin{aligned}\sphericalangle A_1MB_1 &= 180^\circ-\sphericalangle AMA_1-\sphericalangle BMB_1\\ &= 180^\circ-\sphericalangle AMA_1-\sphericalangle AA_1M\\ &= \sphericalangle A_1AM.\end{aligned}$$

Thus $\triangle AMA_1\sim\triangle MB_1A_1$ which implies that $\sphericalangle AA_1M=\sphericalangle MA_1B_1$. Since $\triangle BB_1M\sim\triangle AMA_1\sim\triangle MB_1A_1$, it follows that $\sphericalangle BB_1M=\sphericalangle MB_1A_1$. Hence M is the intersection point of the bisectors of $\sphericalangle AA_1B_1$ and $\sphericalangle BB_1A_1$.

9.3. To prove the right inequality, it is enough to use that the denominators are greater than 1. Hence

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}<a+b+c=1.$$

To show the left inequality, we may assume that $a \le b \le c$. Then

$$\frac{1}{1+bc} \le \frac{1}{1+ca} \le \frac{1}{1+ab}.$$

Applying consecutively the Chebyshev inequality, the Arithmetic mean – Harmonic mean inequality and the well-known inequality $(a+b+c)^2 \ge 3(ab+bc+ca)$ we get that

$$3\left(\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}\right) \ge (a+b+c)\left(\frac{1}{1+bc}+\frac{1}{1+ca}+\frac{1}{1+ab}\right)$$

$$=\frac{1}{1+bc}+\frac{1}{1+ca}+\frac{1}{1+ab} \ge \frac{9}{3+ab+bc+ca} \ge \frac{9}{3+\dfrac{(a+b+c)^2}{3}} = \frac{27}{10}.$$

9.4. It is clear that x and y have different parity. Then $k = x-y$ is an odd number and

$$(3k-6)y^2+(3k^2+10)y+k^3+10k-1=0.$$

The discriminant of this equation is equal to

$$D=-3k^4+24k^3-60k^2+252k+76$$

and must be a perfect square. Since $D = -k^2(k^2-24k+60)+252k+76$, then $D<0$ for $k \le -1$. On the other hand, $D = 3k^3(8-k)+2(38-k^2)+2k(126-29k)$ and therefore $D<0$ for $k \ge 8$. Since $D=-71<0$ for $k=7$, it remains to check the cases $k=1,3,5$. We have $D=289=17^2$, $D=697$ and $D=961=31^2$, respectively, that give the solutions $x=6$, $y=5$ and $x=2$, $y=-3$.

9.5. We shall show that the desired value is equal to 4.

Let (i,j) be the unit square in the i-th row and the j-th column of a 4×4 square. It is easy to check that if the squares $(1,1)$, $(1,2)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,4)$, $(4,3)$, $(4,4)$ are white and the other – black, then the condition of the problem is satisfied.

To prove that for $n \ge 5$ there is no a coloring with the required properties it is enough to deal only with the case $n=5$ (any $n\times n$ square contains a 5×5 square). Consider an arbitrary black-white coloring of a 5×5 square. At least 13 of the unit squares have the same coloring, for example, black. Three cases are possible.

Case 1. One of the rows, say l, contains only black squares. Then one of the other rows contains at least 2 black squares. Hence the corners of the rectangle with two vertices at these squares and the other two in l are black.

Case 2. One of the rows, say l, contains exactly 4 black squares. Then one of the other rows contains at least 3 black squares. Then at least two of them

do not correspond to the white square in l. Hence the corners of the rectangle with two vertices at these squares and the other two in l are black.

Case 3. Any row contains at most 3 black squares. Then at least 3 rows contain exactly 3 black squares, say the rows 1,2,3 from above. We call a column *black* (*white*) if its upper square is black (white). There are exactly 3 black columns. If a row contains 2 black squares that lie in black columns, then obviously there is a rectangle with black corners. Otherwise, two of the black squares in row 2 and two of the black squares in the two white columns are black corners of a rectangle.

The solution is completed.

9.6. a) Let x satisfies the equality $[x]^3+x^2=x^3+[x]^2$. Then setting $t=[x]$ and $\alpha=x-t\in[0,1)$ one has that

$$t^3-t^2=(t+\alpha)^3-(t+\alpha)^2 \iff \alpha(\alpha^2+(3t-1)\alpha+3t^2-2t)=0.$$

Hence either $\alpha=0$, or α is a root of the equation in the brackets. In the second case the discriminant $(3t+1)(1-t)$ of this equation must be non-negative. Since t is an integer, it follows that $t=0$ or $t=1$. Then either $\alpha=0$, $\alpha=1$ or $\alpha=-1$. Now $\alpha\in[0,1)$ implies that $\alpha=0$, i.e., x is an integer.

b) The degree of the polynomial y^3-y^2-1 is odd. Hence this polynomial has a real zero α. Obviously, α is not an integer (in fact, α is unique and $\alpha\in(1,2)$). Then $[\alpha^3]=[\alpha^2+1]=[\alpha^2]+1$ and hence $[\alpha^3]-[\alpha^2]=1=\alpha^3-\alpha^2$.

10.1. The inequality makes sense for $x\in\left(-\infty,-\dfrac{\sqrt{6}}{2}\right]\cup\left[\dfrac{\sqrt{6}}{2},+\infty\right)$. All x in the second interval are solutions of the inequality. Let $x\in\left(-\infty,-\dfrac{\sqrt{6}}{2}\right]$. Then the inequality is equivalent to

$$\sqrt{(x^2-1)(2x^2-3)}>2 \iff 2x^4-5x^2-1>0.$$

Solving the last inequality gives $x\in\left(-\infty,-\dfrac{\sqrt{5+\sqrt{33}}}{2}\right)$.

So, the solutions of the given inequality are

$$x\in\left(-\infty,-\frac{\sqrt{5+\sqrt{33}}}{2}\right)\cup\left[\frac{\sqrt{6}}{2},+\infty\right).$$

10.2. We shall use the standard notation for the elements of $\triangle ABC$.

a) The Cosine theorem for $\triangle AMB$ gives

$$\cos\sphericalangle AMB=\frac{AM^2+BM^2-AB^2}{2AM.MB}.$$

This together with the equality $\sin\sphericalangle AMB = \dfrac{2S}{3AM.MB}$ and the median formula implies that

$$\cot\sphericalangle AMB = \frac{3(AM^2+BM^2-AB^2)}{4S} = \frac{a^2+b^2-5c^2}{12S}.$$

b) It follows as above that

$$\cot\sphericalangle BMC = \frac{b^2+c^2-5a^2}{12S}, \quad \cot\sphericalangle CMA = \frac{c^2+a^2-5b^2}{12S}.$$

Hence

$$\cot\sphericalangle AMB + \cot\sphericalangle BMC + \cot\sphericalangle CMA = -\frac{a^2+b^2+c^2}{4S}.$$

It remains to show that $a^2+b^2+c^2 \geq 4S\sqrt{3}$. The Heron formula and the Arithmetic mean- Geometric mean inequality give

$$S^2 = p(p-a)(p-b)(p-c) \leq p\left(\frac{p-a+p-b+p-c}{3}\right)^3 = \frac{p^4}{27}.$$

Therefore

$$S \leq \frac{p^2}{3\sqrt{3}} = \frac{(a+b+c)^2}{12\sqrt{3}} \leq \frac{a^2+b^2+c^2}{4\sqrt{3}}$$

by the Root mean square inequality.

10.3. It follows by the given condition that $m(m+j-1) = j(j-1)$ and hence $m^2 = (j-m)(j-1)$. If p is a prime divisor of $j-m$ and $j-1$, then it divides m. Therefore p divides j and 1. This contradiction shows that $j-m$ and $j-1$ are coprime. Then $j-m = u^2$ and $j-1 = v^2$, where u and v are non-negative integers. It follows that $uv = m$ and $u^2+uv = j = v^2+1$, and the condition $1 \leq j < 2004$ gives $0 \leq v \leq 44$. So, we have to solve in non-negative integers the equation

$$(*) \qquad u^2 + uv = v^2 + 1.$$

If $v = 0$, then $u = 1$. Assume that the pair $(u_0; v_0)$ is a solution of $(*)$ and $v_0 \geq 1$. Then $u_0 \geq 1$. Since $u_0v_0 \geq 1$, it follows that $u_0 \leq v_0$. Moreover, it is clear that $v_0 < 2u_0$. Set $v_1 = v_0 - u_0$, $0 \leq v_1 < v_0$. We have that $u_0^2 = v_0(v_0-u_0)+1 = (u_0+v_1)v_1+1 = u_0v_1+v_1^2+1$. Set $u_1 = u_0-v_1 = 2u_0-v_0 > 0$. Then

$$(u_1+v_1)^2 = (u_1+v_1)v_1 + v_1^2 + 1 \iff u_1^2 + u_1v_1 = v_1^2 + 1,$$

i.e., we obtain a new solution of $(*)$. If $v_1 = 0$, then $u_1 = 1$. If $v_1 \geq 1$, then $u_1 \geq 1$. Setting $v_2 = v_1 - u_1 < v_1$ and $u_2 = u_1 - v_2$, we get in the same way a new solution. So, we get a sequence of non-negative integers $v_0 > v_1 > \cdots$. If

$v_k = 0$ for some k, then $u_k = 1$ and writing $u_{k-1}, v_{k-1}, \dots, u_0, v_0$, we get the Fibonacci sequence.

Thus solutions of $(*)$ are $(u;v) = (1;0)$, $(1;1)$, $(2;3)$, $(5;8)$, $(13;21)$, and for the other solutions we have that $v > 44$. The first solution of $(*)$ gives $j = v^2 + 1 = 0^2 + 1 = 1$ and then $m = uv = 0$, which does not satisfy the given condition. The other solutions give the following solutions of the problem: $j = 1^2 + 1 = 2$, $j = 3^2 + 1 = 10$, $j = 8^2 + 1 = 65$ and $j = 21^2 + 1 = 442$.

10.4. a) It follows by a direct verification.

б) Writing the equation in the form

$$(x+a)((a^2+4a+2)x^2 + (1-a)x + a) = 0, \tag{3}$$

we get that $a < 0$. Moreover, the quadratic polynomial in (3) must have two distinct real zeros, i.e. $D = (1-a)^2 - 4a(a^2+4a+2) > 0 \iff (a+1)(-4a^2 - 11a + 1) > 0$. Solving this inequality and having in mind that $a < 0$, we get that

$$a \in \left(-\infty, \frac{-11-\sqrt{137}}{8}\right) \cup (-1, 0). \tag{4}$$

The roots of the quadratic polynomial in (1) are positive if and only if $\dfrac{a}{a^2+4a+2} > 0$ and $-\dfrac{1-a}{a^2+4a+2} > 0$. Then $a \in (-2-\sqrt{2}, -2+\sqrt{2})$ and using (4), we obtain that

$$a \in \left(-2-\sqrt{2}, \frac{-11-\sqrt{137}}{8}\right) \cup \left(-1, -2+\sqrt{2}\right). \tag{5}$$

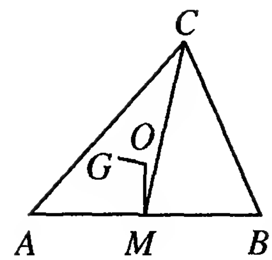
It remains to see when $-a$ is a zero of the quadratic polynomial in (3). We have that

$$(a^2+4a+2)(-a)^2 + (1-a)(-a) + a = 0 \iff a^2(a^2+4a+3) = 0,$$

and hence $a = -3, -1, 0$. So, the answer of the problem is

$$a \in \left(-2-\sqrt{2}, -3\right) \cup \left(-3, \frac{-11-\sqrt{137}}{8}\right) \cup \left(-1, -2+\sqrt{2}\right).$$

10.5. Set $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$. We have that $\overrightarrow{OM} = \frac{1}{2}(\vec{a}+\vec{b})$ and hence



$$\overrightarrow{OG} = \frac{1}{6}(3\vec{a}+\vec{b}+2\vec{c}).$$

On the other hand, $\overrightarrow{CM} = \frac{1}{2}(\vec{a}+\vec{b}-2\vec{c})$. Then

$$\begin{aligned} 0 &= \overrightarrow{OG}.\overrightarrow{CM} = (3\vec{a}+\vec{b}+2\vec{c})(\vec{a}+\vec{b}-2\vec{c}) \\ &= 3R^2 + 3\vec{a}\vec{b} - 6\vec{a}\vec{c} + \vec{a}\vec{b} + R^2 - 2\vec{b}\vec{c} + 2\vec{a}\vec{c} + 2\vec{b}\vec{c} - 4R^2 \end{aligned}$$

and therefore $0 = 4\vec{a}(\vec{b}-\vec{c})$. Hence $OA \perp BC$, i.e. $AB = AC$.

10.6. Denote by V and E the sets of the vertices and the edges of G, respectively. For any vertex $x \in V$, let $\Gamma(x)$ be the set of the edges of G, which are adjacent to x and let $d(x) = |\Gamma(x)|$. Then for $x, y \in V$ one has that

$$|\Gamma(x) \cap \Gamma(y)| = |\Gamma(x)| + |\Gamma(y)| - |\Gamma(x) \cup \Gamma(y)| \geq d(x) + d(y) - |V|.$$

Summing up these inequalities for all the edges $(x,y) \in E$, we get that

$$\begin{aligned} 3t(G) &= \sum_{(x,y)\in E} |\Gamma(x)\cap\Gamma(y)| \geq \sum_{(x,y)\in E} (d(x)+d(y)) - |V|.|E| \\ &= \sum_{x\in V} d^2(x) - |V|.|E| \end{aligned}$$

(here $t(G)$ is the number of the triangles in G). Hence

$$3t(G) \geq \frac{1}{|V|}\left(\sum_{x\in V} d(x)\right)^2 - |V|.|E| = \frac{4|E|^2}{|V|} - |V|.|E|.$$

In our case we have that $|V| = 10$ and $|E| = 26$. Therefore $t(G) \geq \frac{52}{15}$ and hence $t(G) \geq 4$.

11.1. The numbers form a geometric progression if and only if

$$2^{\sin x}.2^{\cos x} = (2 - 2^{\sin x + \cos x})^2.$$

This is equivalent to $4^{\sin x+\cos x} - 5.2^{\sin x+\cos x} + 4 = 0$. Setting $y = 2^{\sin x+\cos x}$ gives $y^2 - 5y + 4 = 0$, i.e., $y = 4$ or $y_2 = 1$.

If $2^{\sin x+\cos x} = 4$, then $\sin x + \cos x = 2$, i.e., $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}$, which is impossible.

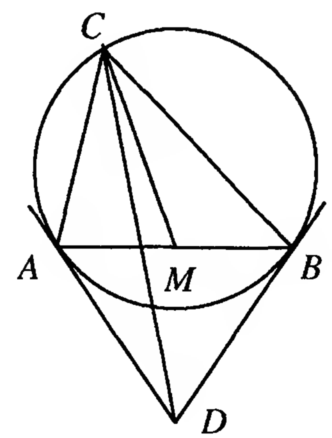
If $2^{\sin x+\cos x} = 1$, then $\sin x + \cos x = 0$, i.e., $\sin\left(x + \frac{\pi}{4}\right) = 0$. Hence $x = -\frac{\pi}{4} + k\pi$, where k is an integer. Since $x \in (-\pi, \pi)$, we get finally that $x = -\dfrac{\pi}{4}$ and $x = \dfrac{3\pi}{4}$.

11.2. The Sine theorem for $\triangle AMC$ and $\triangle BMC$ gives

$$\frac{AM}{CM} = \frac{\sin \measuredangle ACM}{\sin\alpha}, \quad \frac{BM}{CM} = \frac{\sin(\gamma - \measuredangle ACM)}{\sin\beta}.$$

Since $AM = BM$, we get that $\dfrac{\sin \measuredangle ACM}{\sin\alpha} = \dfrac{\sin(\gamma - \measuredangle ACM)}{\sin\beta}$ and therefore

$$(1) \qquad \tan \measuredangle ACM = \frac{\sin\alpha\sin\gamma}{\sin\beta + \sin\alpha\cos\gamma}.$$



The Sine theorem for $\triangle ADC$ and $\triangle BDC$ implies that

$$\begin{aligned} \frac{AD}{CD} &= \frac{\sin(\gamma - \measuredangle BCD)}{\sin(\alpha+\gamma)}, \\ \frac{BD}{CD} &= \frac{\sin \measuredangle BCD}{\sin(\beta+\gamma)}. \end{aligned}$$

Then the equality $AD = BD$ shows that $\dfrac{\sin(\gamma - \measuredangle BCD)}{\sin\beta} = \dfrac{\sin \measuredangle BCD}{\sin\alpha}$, i.e.

$$(2) \qquad \tan \measuredangle BCD = \frac{\sin\alpha\sin\gamma}{\sin\beta + \sin\alpha\cos\gamma}.$$

It follows by (1) and (2) that $\tan \measuredangle ACM = \tan \measuredangle BCD$. Since these angles are acute, they are equal.

11.3. Consider a group with maximal number of people such that any two of them are not familiar. It is clear that if there are l people in this group, then $l \le m-1$. Moreover, the maximality of l implies that any of the other $N - l$ people is familiar to at least one of the l people in the group. Hence some of these l people is familiar to at least $\dfrac{N-l}{l}$ people. Since

$$\frac{N-l}{l} = \frac{N}{l} - 1 \ge \frac{N}{m-1} - 1 = n + \frac{1}{m-1} - 1 > n - 1,$$

there is a person who is familiar to n people.

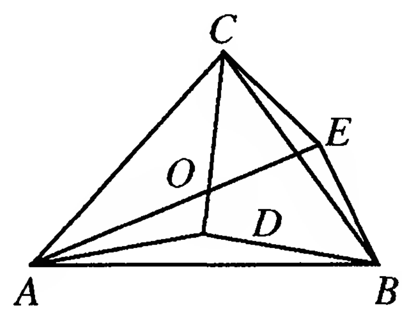
Let $N = mn - n = (m-1)n$ and consider $m-1$ groups by n people such that any two people from one group are familiar and there are no familiar people from different groups. Then among any m people there are two from one and the same group, i.e. they are familiar. On the other hand, any of the

people is familiar to $n-1$ of the other and hence the statement is not true if $N < mn - n + 1$.

11.4. Since the isosceles triangles ABD and CBE are similar, we have $\dfrac{AB}{DB} = \dfrac{CB}{BE}$. Using that $\sphericalangle ABC = \sphericalangle DBE$, we get that $\triangle ABC \sim \triangle DBE$ and $\dfrac{DB}{AB} = 2\cos\varphi$, where $\varphi = \sphericalangle ABD$. Hence $\sphericalangle ACB = \sphericalangle DEB = \gamma$.

Then

$$\begin{aligned} S_{ACE} &= \frac{1}{2} AC.CE.\sin(\varphi+\gamma) \\ &= \frac{1}{2} b \cdot \frac{a}{2\cos\varphi} \cdot \sin(\varphi+\gamma) \end{aligned}$$



and

$$\begin{aligned} S_{EBDC} &= \frac{1}{2} CB.DE.\sin(\varphi+\gamma) \\ &= \frac{1}{2} a \cdot \frac{b}{2\cos\varphi} \cdot \sin(\varphi+\gamma). \end{aligned}$$

Hence $S_{ACE} = S_{EBDC}$, which implies the statement of the problem.

11.5. We consider two cases.

Case 1. Let $(a,b) = (a,c) = 1$. Then $(a,bc) = 1$ and hence there are integers u and v such that $ua + vbc = 1$. This means that a divides $-vbc + 1$. If $k \geq 1$ is a positive integer such that a divides $-v-k$, then a divides $kbc+1$, i.e. $kbc + 1 = at$. Hence setting $x = 2^t$, $y = 2^{kc}$ and $z = 2^{kb}$ we have that

$$y^b + z^c = 2^{kbc} + 2^{kbc} = 2^{kbc+1} = \left(2^t\right)^a = x^a.$$

Case 2. Let $(c,a) = (c,b) = 1$. Then $(c,ab) = 1$ and as above we find a positive integer k such that c divides $kab+1$, i.e., $kab + 1 = ct$. Hence setting $x = 2(2^a-1)^{kb}$, $y = (2^a-1)^{ka}$ and $z = (2^a-1)^t$ one has that

$$x^a - y^b = 2^a(2^a-1)^{kab} - (2^a-1)^{kab} = (2^a-1)^{kab+1} = z^c.$$

11.6. a) At each step the triangle from which we choose a point, is divided into three new triangles, i.e. the numbers of the triangles increases by two. Hence at the n-th step we have $2n+1$ triangles.

b) We shall prove by induction on n that removing any triangle then there is a pairing of the remaining triangles such that the triangles in any pair have a common side.

The statement is trivial for $n = 1$. Assume that it is true for $n = k$. We shall prove it for $n = k+1$. Let a point O in $\triangle MNP$ is added at the $(k+1)$-th step. Remove any $\triangle XYZ$. If it is some of $\triangle OMN$, $\triangle OMP$ or $\triangle ONP$ (we

may assume $\triangle OMN$), we consider the configuration obtained by removing $\triangle MNP$ at the k-th step.

By the induction hypothesis, the remaining triangles can be paired in such a way that the triangles in any pair have a common side. Adding the pair $(\triangle OMP, \triangle ONP)$ we obtain the desired pairing.

If $\triangle XYZ$ does not coincide with $\triangle OMN$, $\triangle OMP$ and $\triangle ONP$, we consider the configuration obtained by removing $\triangle XYZ$ at the k-th step. Assume that $\triangle MNP$ is paired with $\triangle QMN$. Then replacing the pair $(\triangle MNP, \triangle QMN)$ with the pairs $(\triangle OMN, \triangle QMN)$ and $(\triangle OMP, \triangle OPN)$ completes the induction.

Since the area of $\triangle ABC$ equals 1, then the minimal area of a triangle does not exceed $\dfrac{1}{2n+1}$. Remove a triangle of minimal area. Then as we proved above the remaining triangles can be paired such that the triangles in any pair have a common side. Since the number of the pairs is equal to n, there is a pair of total area at least $\dfrac{1-\frac{1}{2n+1}}{n} = \dfrac{2}{2n+1}$.

12.1. It is easy to see that $a, c \geq 0$. Since 3^c is congruent to 1 or 3 modulo 8, then $0 \leq a \leq 2$. If $a = 0$ or $a = 1$, then 2 divides 3^c or 8 divides $3^c + 1$, a contradiction.

Let $a = 2$, i.e., $8b^2 - 3^c = 279$. The cases $c = 0, 1$ are impossible and hence $c \geq 2$. Then 3 divides b and setting $b = 3d$ gives $8d^2 - 3^{c-2} = 31$. If $c \geq 3$, then 3 divides $d^2 + 1$, a contradiction. Therefore $c = 2$ and $d = \pm 2$, i.e. $a = 2$, $b = \pm 6$ and $c = 2$.

12.2. Since the denominator of the function is positive, the given condition means that $ax - 1 \leq x^4 - x^2 + 1$ for any x, and that the equality is attained for some x.

Let $a \geq 0$. Then $ax \leq 0$ for $x \leq 0$ and hence a is the minimum of the function $g(x) = \dfrac{x^4 - x^2 + 2}{x}$ for $x > 0$. Since $g'(x) = \dfrac{(3x^2+2)(x+1)(x-1)}{x^2}$, it follows that this minimum is equal to $g(1) = 2$. The case $a \leq 0$ can be reduced to the previous one replacing a by $-a$ and x by $-x$. So $a = \pm 2$.

12.3. Let a plane π bisect the volume of $ABCD$ and meet the edges AB, BC, CD and DA at points M, Q, N and P, respectively. Set $x = \dfrac{AM}{BM}$, $y = \dfrac{CN}{DN}$, $z = \dfrac{AP}{DP}$ and $t = \dfrac{CQ}{BQ}$. If $T = \pi \cap AC$ (we assume $T = \infty$ if $\pi || AC$), then the Menelaus theorem implies that $\dfrac{x}{t} = \dfrac{AT}{CT} = \dfrac{z}{y}$, i.e., $xy = zt$.

On the other hand,

$$\begin{aligned}\frac{1}{2} &= \frac{V_{AMQCNP}}{V_{ABCD}} = \frac{V_{AMQCP}+V_{QCNP}}{V_{ABCD}} = \frac{S_{AMQC}}{S_{ABC}}.\frac{AP}{AD} + \frac{S_{QCN}}{S_{BCD}}.\frac{DP}{AD}\\ &= \left(1-\frac{BM}{AB}.\frac{BQ}{BC}\right)\frac{AP}{AD} + \frac{CN}{CD}.\frac{CQ}{CB}.\frac{DP}{AD}\\ &= \left(1-\frac{1}{(1+x)(1+t)}\right)\frac{z}{1+z} + \frac{yt}{(1+y)(1+t)(1+z)},\end{aligned}$$

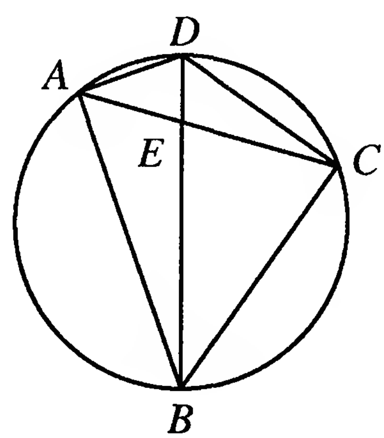
and hence

(1) $\quad 2z(1+y)(x+t+xt) + 2yt(1+x) = (1+x)(1+y)(1+z)(1+t).$

Let now the plane from the condition of the problem meet the edge BC in an interior point. Then it meets the edge AD. Since $x = y$ and $xy = zt$, then (1) easily implies that $(x-1)(t^2+t(x+1)^2+x^2) = 0$. But $x \neq 1$ and the second factor is positive. Hence this case is impossible.

Thus the given plane meets the edges AC and BD. Replacing C by D and D by C, we get $xy = 1 = zt$ and the inequality still holds. Then $(x-1)(t^2-1) = 0$, i.e. $t = z = 1$, which completes the solution.

12.4. If the perpendicular bisector of CD meets BD at point O, then



$$\begin{aligned}\sphericalangle COD &= 180^\circ - 2\sphericalangle ODC\\ &= 180^\circ - 2\sphericalangle BAC\\ &= \sphericalangle ACB = \sphericalangle ADO\end{aligned}$$

and therefore $AD \| CO$. Hence $\frac{OE}{3} = \frac{CO}{AD} = \frac{OE+3}{5}$, which implies that $OE = \frac{9}{2}$. This shows that O is the midpoint of BD and then $\sphericalangle BCD = 90^\circ$.

12.5. a) Let $a_1 < \cdots < a_{n-1} < a_n = 2000 < a_{n+1} < \cdots < a_m$ be the elements of a "good" set. Since $a_{i+1} \geq 2a_i$, then $2\ 000\ 000 > a_m \geq 2^{m-n}2000$ and hence $m - n \leq 9$.

On the other hand, the equality $2000 = 2^4 5^3$ shows that $a_i = 2^{k_i}5^{l_i}$ for $i \leq n-1$, where $0 \leq k_i \leq k_{i+1} \leq 4$, $0 \leq l_i \leq l_{i+1} \leq 3$ and $k_i + l_i \leq 6$. Hence $n \leq 8$ and so A has at most $8+9 = 17$ elements. An example of a "good" set of 17 elements is obtained by setting $a_i = 2^{i-1}$, $1 \leq i \leq 5$, $a_i = 2^4 5^{i-5}$, $6 \leq i \leq 8$, $a_i = 2^{i-4}5^3$, $9 \leq i \leq 17$.

b) For a "good" set of maximal cardinality one has that $m = 17$ and $n = 8$, i.e. $a_8 = 2000$. Moreover, $k_i + l_i = i - 1$ for $1 \leq i \leq 7$, which shows that $a_1 = 1$ and that the subset $\{a_2, \ldots, a_7\}$ is determined by the numbers $1 \leq i_1 < i_2 < i_3 \leq 7$ such that $l_{i_1} = 0$, $l_{i_1+1} = 1$, $l_{i_2} = 1$, $l_{i_2+1} = 2$, $l_{i_3} = 2$ and $l_{i_3+1} = 3$.

There are $\binom{7}{3} = 35$ possibilities for this subset. Since $2^9 < 2^8.3 < 1000 < 2^{10}$, it follows that either $a_i = 2^{i-4}5^3$ for $9 \leq i \leq 17$, or there is an index j, $9 \leq j \leq 17$ such that $a_i = 2^{i-4}5^3$ for $8 \leq i < j$ and $a_i = 2^{i-5}5^3 3$ for $j \leq i \leq 17$. Hence there are 10 possibilities for the subset $\{a_9, \dots, a_{17}\}$. So, the number of the the "good" sets of maximal cardinality equals $35.10 = 350$.

12.6. Set $R(x) = P(x)P(x+1)\dots P(x+2003)$. It follows by the given condition that if x is greater than the largest real root of $P(x)$, then

$$\frac{Q(x)}{R(x)} = \frac{Q(x+1)}{R(x+1)}. \tag{1}$$

We get by induction that $\dfrac{Q(x)}{R(x)} = \dfrac{Q(x+n)}{R(x+n)}$ for any positive integer n. Note that $\lim\limits_{n\to\infty} \dfrac{Q(x+n)}{R(x+n)}$ is either a finite number, independent of x, or ∞. On the other hand, this limit is equal to $\dfrac{Q(x)}{R(x)}$. Hence $Q(x) = cR(x)$ for any x, where $c \neq 0$ is a constant.

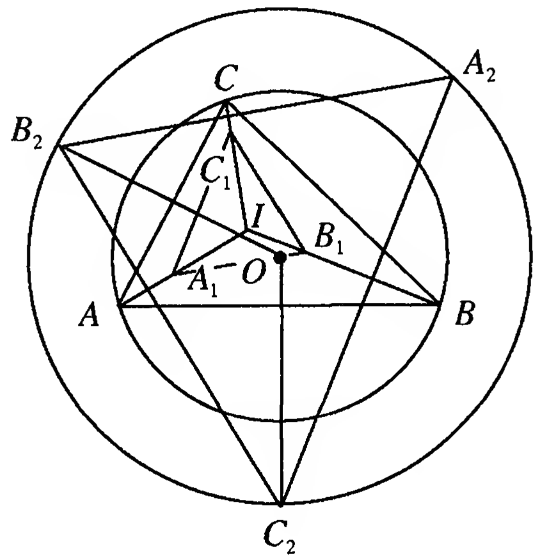
Conversely, it is clear that if $Q(x) = cP(x)P(x+1)\dots P(x+2003)$, then the condition of the problem is satisfied.

Remark. Getting equality (1), the solution can be completed by comparing the coefficient of the respective polynomials.

53. Bulgarian Mathematical Olympiad

National Round

1. Let O be the circumcenter of $\triangle ABC$. Suppose that O coincides with the circumcenter of $\triangle A_2B_2C_2$.



Then $\sphericalangle C_2OB_2 = 2\sphericalangle C_2A_2B_2 = 2(180^\circ - \sphericalangle BIC) = \sphericalangle B + \sphericalangle C$. Consequently,

$$\sphericalangle OB_2C_2 = \frac{1}{2}(180^\circ - \sphericalangle C_2OB_2) = \frac{\sphericalangle A}{2} = \sphericalangle IAC.$$

Thus $OB_2 \perp AC$, i.e. OB_2 is the perpendicular bisector of AC. Then the points A, A_1, C_1 and C lie on a circle with center B_2. It follows that $\sphericalangle A_1C_1I = \sphericalangle A_1AC = \frac{\sphericalangle A}{2}$ and $\sphericalangle C_1A_1I = \frac{\sphericalangle C}{2}$. Analogously $\sphericalangle B_1A_1I = \frac{\sphericalangle B}{2}$. Hence

$$\sphericalangle IC_1A_1 + \sphericalangle C_1A_1B_1 = \frac{\sphericalangle A + \sphericalangle B + \sphericalangle C}{2} = 90^\circ,$$

i.e., $C_1I \perp A_1B_1$. Analogously $B_1I \perp A_1C_1$ and $A_1I \perp B_1C_1$, i.e., I is the orthocenter of $\triangle A_1B_1C_1$.

Conversely, let I be the orthocenter of $\triangle A_1B_1C_1$. Then $\sphericalangle B_1A_1C_1 = 180^\circ - \sphericalangle B_1IC_1 = \frac{\sphericalangle B}{2} + \frac{\sphericalangle C}{2}$. Consequently $\sphericalangle A_1C_1I = 90^\circ - \sphericalangle B_1A_1C_1 = \frac{\sphericalangle A}{2}$. Then the quadrilateral AA_1C_1C is cyclic and therefore B_2 lies on the perpendicular bisector of AC.

Let O be the circumcenter of $\triangle ABC$. Since O lies on the perpendicular bisector of AC we conclude that $\sphericalangle OB_2C_2 = \sphericalangle CAI = \frac{\sphericalangle A}{2}$. Analogously, $\sphericalangle OC_2B_2 = \sphericalangle BAI = \frac{\sphericalangle A}{2}$ and therefore $OB_2 = OC_2$.

We prove in the same way that $OA_2 = OC_2$ and therefore O is the circumcenter of $\triangle A_2B_2C_2$.

2. a) Set $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. We have that $S_2 = \frac{3}{2}$, $S_7 = \frac{3\,121}{140}$,

$$S_{22} - S_7 = \frac{1}{8} + \frac{1}{22} + \frac{1}{10} + \frac{1}{20} + \frac{1}{11} + \frac{1}{19} + \frac{1}{14} + \frac{1}{16} + \frac{1}{13} + \frac{1}{17} +$$

$$\frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \frac{1}{21} = \frac{30a}{b} + \frac{51}{140} = \frac{3c}{d},$$

where $(a,b) = (c,d) = 1$. It is easy to see that $a \equiv b \pmod 3$. Hence $3 \nmid c, d$, $c \not\equiv d \pmod 3$, $p_{22} = 3p'_{22}$ and $3 \not| p'_{22}$. Similarly, we have that $S_{67} - S_{22} = \frac{90e}{f} + \frac{c}{d}$, where $3 \nmid f$. It follows that $3 \nmid p_{67}, q_{67}$.

б) Set $S_n = \frac{k_n}{3^{m_n} l_n}$, where $3 \nmid k_n, l_n$. Then

$$\begin{aligned} S_{3n} &= \frac{S_n}{3} + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \cdots + + \frac{1}{3n-2} + \frac{1}{3n-1} \\ &= \frac{k_n}{3^{m_n+1} l_n} + 3.\frac{a_n}{b_n} = \frac{k_n b_n + 3^{m_n+2} l_n a_n}{3^{m_n+1} l_n b_n}, \end{aligned}$$

where $3 \nmid b_n$. Therefore, if $m_n \geq -1$, then $m_{3n} = m_n + 1$. Analogously, we have that $m_{3n+2} = m_n + 1$ for $m_n \geq -1$ and $m_{3n+1} = m_n + 1$ for $m_n \geq 0$. Since $m_1 = 0$, $m_2 = m_7 = m_{22} = -1$ and $m_{67} = 0$, it is easy to see that the answer is $n = 2, 7, 22$.

3. Consider a graph G with n vertices corresponding to the tourists and two vertices connected when they are familiar.

The first condition of the problem means that there is no triangle in G.

The second condition of the problem means that there is a cycle of odd length in the graph. Indeed, if all cycles are of even length, then the vertices can be partitioned into two groups such that there is no edge in any of them.

Let $A_1, A_2, \ldots, A_k$ be a cycle with minimal odd length in G. Since there is no a triangle in G, and because of minimality, it follows that every vertex outside this cycle is connected with at most two vertices from the cycle. Hence the number of edges of the form (X, A_i), $X \neq A_j$, $j = 1, 2, \ldots, k$ does not exceed $2(n-k)$. Denote by $d(A_i)$ the degree of the vertex A_i and set $\delta = \min\limits_{1 \leq i \leq k} d(A_i)$. Obviously $\sum\limits_{i=1}^{k} d(A_i) = |E^*| + 2k$, where E^* is the set of edges XA_i. We have that

$$2(n-k) \geq |E^*| = \sum_{i=1}^{k} d(A_i) - 2k \geq k\delta - 2k.$$

Hence $\delta \leq \frac{2n}{k}$ and $k \geq 5$ implies that $\delta \leq \frac{2n}{5}$.

4. The answer is *no*.

We shall prove that applying any of the given changes the number of a's in odd (even) positions does not change its parity. Indeed, let the replacement $aba \to b$ be applied to the word $w_1 abaw_2$. In the new word $w_1 bw_2$ all a's in w_1 do not change their positions and all a's in w_2 shift by two position to the left. Hence such a's keep the parity of their positions. Deleting both a's in aba decreases the number of a's in odd (even) positions by two.

Analogously, the same is true by applying the operation $bba \to a$ to the word $w_1 bbaw_2$.

Since the replacements $a \to aba$ and $a \to bba$ are converse to the above, they have the same property.

It remains to observe that the number of a's in even positions in the words $b\underbrace{aa\ldots a}_{2003}$ and $\underbrace{aa\ldots a}_{2003}b$ is 1002 and 1001, respectively.

5. The answer is 1, 7 or 49.

Suppose first that $ad \neq bc$. The set of points $(ax+by, cx+dy)$, $x, y \in (0,1)$, coincides with the interior of the parallelogram with vertices $A = (0,0)$, $B = (a,c)$, $C = (b,d)$ and $D = (a+b, c+d)$. Its area S equals $|ad - bc|$. The Pick formula implies that $S = n + \frac{m}{2} - 1$, where n (respectively m) denotes the number of lattice points (i.e., with integer coordinates) in the interior (respectively on the boundary) of the parallelogram. Set $e = (a,c)$, $f = (b,d)$, $a = ea_1$, $c = ec_1$, $b = fb_1$ and $d = fd_1$. The interior points of the side AB have coordinates (ax, cx), $x \in (0,1)$. Thus the number of such lattice points is $e-1$. Analogously, the number of the interior lattice points on BD, CD and AC equals $f-1$, $e-1$ and $f-1$, respectively. Consequently $\frac{m}{2} = e+f$ and the first condition of the problem can be written as

$$(1) \qquad ef|a_1 d_1 - b_1 c_1| = 2003 + e + f.$$

Since $e = (a,c) = 6$, it follows that f divides $2009 = 7^2.41$ and $6f$ divides $2009 + f$. This is possible only for $f = 1, 7, 49$. For each of these values of f the numbers $e = 6$, $a_1 = 1 + \frac{2009+f}{6f}$, $b_1 = c_1 = d_1 = 1$ satisfy (1) and hence $(b,d) = 1, 7$ or 49.

Suppose now that $ad = bc$. It is easy to see that $a_1 = b_1$ and $c_1 = d_1$. For $x \in \left(0, \frac{1}{e}\right)$ set $y = \frac{1-xe}{f} < \frac{1}{f}$. Then $ax + by = a_1$ and $cx + dy = c_1$ are integers, and $ex + fy = 1$. It follows that in this case there are infinitely many pairs (x,y) satisfying (1), a contradiction.

6. a) Let $t \in \{0, 1, 2, \ldots, p-1\}$. Consider the remainders of $t - a_i$, $1 \le i \le m$ and b_j, $1 \le j \le n$, modulo p. Their number is $m + n > p$ and hence two of them are equal. Since the remainders of $t-a_i$ and $t-a_j$, b_i and b_j, respectively, $i \neq j$, are different, it follows that $t - a_r \equiv b_s \pmod p$, i.e., $a_r + b_s \equiv t \pmod p$ for some r and s. Since t is an arbitrary remainder modulo p, we conclude that $k = p$.

b) Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. For any two sets X and Y denote $X + Y = \{x + y \pmod p) | x \in X, y \in Y\}$. We have to prove that $k = |A + B| \geq m + n - 1$. To do this, we may assume that $m \leq n$ and we shall use induction on m.

For $m = 1$ and any n the statement is true, since $a_1 + b_i \neq a_1 + b_j \pmod p$ if $i \neq j$ and $|a_1 + B| = |B| = n = 1 + n - 1$.

Suppose that the statement is true for any two sets X and Y such that $|X| < m$, $|X| < |Y|$ and $|X| + |Y| \leq p$. Let $|A| = m > 1$ and $|B| = n$, where $m \leq n$ and $m + n \leq p$. Then $n < p$ and hence there exists $c \notin B$. Take different $a_1, a_2 \in A$. As the sequence $c + t(a_2 - a_1) \pmod p$, $t = 1, 2, \dots, p-1$, contains all remainders except c, then $b = c + t(a_2 - a_1) \in B$ for some t. Let t be the minimal number with this property. The set $A' = \{b - a_2\} + A$ contains the elements $b - a_2 + a_1$ and $b - a_2 + a_2 = b$. Note that $b - a_2 + a_1 = c + (t-1)(a_2 - a_1) \notin B$. Since $|A' + B| = |\{b - a_2\} + A + B|$, it is enough to prove that $|A' + B| \geq m + n - 1$.

Set $F = A' \cap B$ and $G = A' \cup B$. Since $b \in F$, $b - a_2 + a_1 \notin F$ and $b - a_2 + a_1 \in A'$, then F is a proper non-empty subset of A'. So B is a proper subset of G. It follows that $0 < |F| < m \leq n < |G|$. On the other hand, $m + n = |A'| + |B| = |A' \cap B| + |A' \cup B| = |F| + |G|$. Note also that $F + G \subset A' + B$ (for $f \in F$ and $g \in G$, we may assume that $g \in A'$ and then $f \in F \subset B$ implies that $f + g \in A' + B$). Thus $|A'| + |B| \geq |F| + |G|$. Then the inequalities $0 < |F| < m \leq n < |G|$, $|F| + |G| \leq p$ and the induction hypotheses imply that the statement is true for the sets F and G. Hence

$$|A + B| = |A' + B| \geq |F + G| \geq |F| + |G| - 1 = |A'| + |B| - 1 = m + n - 1$$

which completes the induction.

Remark. This problem is known as the Cauchy-Davenport theorem.

Team selection test for 20. BMO

1. There exists. Let us take $a_0 = 1$, $a_i = 2004!a_0a_1\dots a_{i-1} - 1$, $i \ge 1$ and $A_i = \{2, 3, \dots, 2004, a_0, a_1, \dots, a_i\}$, $i \ge 0$. Then

$$\begin{aligned}&\left(\prod\nolimits_{a\in A_{i-1}} a - \sum\nolimits_{a\in A_{i-1}} a^2\right) - \left(\prod\nolimits_{a\in A_i} a - \sum\nolimits_{a\in A_i} a^2\right) - 1\\ &= a_i^2 - 1 - (a_i - 1)\prod_{a\in A_{i-1}}\\ &= (a_i - 1)(a_i + 1 - 2004!a_0a_1\dots a_{i-1}) = 0.\end{aligned}$$

Hence

$$\prod_{a\in A_n} a = \sum_{a\in A_n} a^2 \text{ for } n = \prod_{a\in A_0} a - \sum_{a\in A_0} a^2.$$

2. The Arithmetic mean – Geometric mean inequality (for any $k = 2, 3, \dots, n$ implies that

$$ka_kc_k = ka_kb_k^{\frac{1}{k}}\underbrace{c_{k-1}^{\frac{1}{k}}\dots c_{k-1}^{\frac{1}{k}}}_{k-1 \text{ times}} \le a_k^kb_k + (k-1)c_{k-1}.$$

Summing up these inequalities and adding the equality $a_1c_1 = a_1b_1$ gives the desired inequality.

3. Define an oriented graph G with vertices the elements of A and oriented edge xy if $f(x) = y$. We have to count the graphs G such that:

– there are no cycles with length greater than 1;

– there is a chain $a_2\dots a_n$ with length $n-2$ and there is no chain with length $n-1$;

– the only edge outside this chain has the form a_1a_j, where $3 \le j \le n$;

– there is a unique loop a_na_n.

The chain can be chosen in $n!$ ways, and the edge outside it – in $n-2$ ways. Note that the graphs for which this edge is a_1a_3, are counted two times and hence their number is equal to $\binom{n}{2}(n-2)!$. So the answer of the problem is

$$n!(n-2) - \binom{n}{2}(n-2)! = \frac{n!(2n-5)}{2}.$$

4. Since the polygon A is convex, the sum of the projections of its sides on the line A_iA_{i+1} is $2p_i$. Consider the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_2A_3}, \dots, \overrightarrow{A_nA_1}$ and their opposite vectors. Put at the end of the vector $\overrightarrow{A_1A_2}$ the vector forming minimal positive angle with this vector (if these vectors are two we choose that of the form $\overrightarrow{A_iA_{i+1}}$). Do the same with the new vector, etc.

In this way we get a convex polygon $B = B_1B_2\dots B_{2n}$ with equal and parallel opposite sides. Hence its main diagonals have a common point, say O.

Let q_i be the length of the projection of B on the line B_iB_{i+1}. Considering the rectangle with sidelengths q_i and $\mathrm{dist}(B_iB_{i+1}, B_{i+n}B_{i+n+1})$, containing B, we get that

$$\frac{|B_iB_{i+1}|}{q_i} \leq 4\frac{S_{\triangle B_iB_{i+1}O}}{S_B}.$$

So

$$\sum_{i=1}^{n} \frac{|A_iA_{i+1}|}{p_i} = \sum_{i=1}^{2n} \frac{|B_iB_{i+1}|}{q_i} \leq 4.$$

The equality is attained if and only if B is a rectangle, i.e., A is a rectangle.

5. Considering the polynomials $p(x) = x^{m-1} + x^{m-2} + \cdots + x + 1$ and $q(x) = x^{m-1} + x^{m-2} + \cdots + x + a$, $a \neq 1$, shows that the desired minimal numbers does not exceed $2m-2$ (one has that $f(u,v) = (a-1)(u^{m-1} + u^{m-2} + \cdots + u) + (1-a)(v^{m-1} + v^{m-2} + \cdots + v)$). We shall prove by induction on m that the number of the non-zero coefficient is at least $2m-2$.

If $p(x)$ or $q(x)$ contains a monomial which does not appear in the other polynomial, then the non-zero coefficient in $f(u,v)$ are at least $2m$. So we may assume that $p(x)$ and $q(x)$ contain the same monomials. Note also that multiplying some of $p(x)$ and $q(x)$ by non-zero number does not change the non-zero coefficient of $f(u,v)$.

For $m = 2$ one has that $p(x) = ax^n + bx^k$, $q(x) = cx^n + dx^k$ and $ad - bc \neq 0$. Then $f(u,v) = (ad-bc)u^nv^k + (bc-ad)u^kv^n$ has exactly two non-zero coefficients. Let $m = 3$ and let $p(x) = x^k + ax^n + bx^\ell$, $q(x) = x^k + cx^n + dx^\ell$ and $ad - bc \neq 0$. Then

$$\begin{aligned} f(u,v) &= (ad-bc)u^\ell v^k + (bc-ad)u^\ell v^n + (c-a)u^kv^n + (a-c)u^nv^k + \\ &\quad +(d-b)u^kv^\ell + (b-d)u^\ell v^k. \end{aligned}$$

The first two coefficients are non-zero. Since the equalities $a = c$ and $b = d$ do not hold simultaneously, then at least two of the last four coefficients are also non-zero.

Let now $m \geq 4$ and let $p(x) = p_1(x) + ax^n + bx^k$, $q(x) = q_1(x) + cx^n + dx^k$, $ad - bc \neq 0$ and any of the polynomials $p_1(x)$ and $q_1(x)$ has $m-2 \geq 2$ non-zero coefficients. Then $f(u,v) = f_1(u,v) + f_2(u,v) + f_3(u,v)$, where

$$\begin{aligned} f_1(u,v) &= p_1(u)q_1(v) - p_1(v)q_1(u), \\ f_2(u,v) &= (au^n + bu^k)q_1(v) + (cv^n + dv^k)p_1(u) \\ &\quad -(av^n + bv^k)q_1(u) - (cu^n + du^k)p_1(v), \\ f_3(u,v) &= (ad-bc)u^nv^k + (bc-ad)u^kv^n, \end{aligned}$$

and the different polynomials have no similar monomials. If $p_1(x) \neq \alpha q_1(x)$, then, by the induction hypothesis, $f_1(u,v)$ has at least $2(m-2) - 2 = 2m-6$

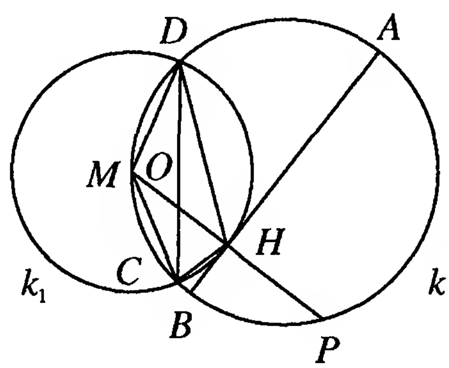
non-zero coefficients. Moreover, $f_2(u,v)$ has at leas two non-zero coefficients and $f_3(u,v)$ has two non-zero coefficients.

If $p_1(x) = \alpha q_1(x)$, $\alpha \neq 0$, then

$$f_2(u,v) = q_1(v)\left[(a - c\alpha)u^n + (b - d\alpha)u^k\right] + q_1(u)\left[(c\alpha - a)v^n + (d\alpha - b)v^k\right].$$

Since the equalities $a - c\alpha = 0$ and $b - d\alpha = 0$ do not holds simultaneously, the polynomial $f_2(u,v)$ has at least $2m-2$ non-zero coefficients (two times more than these of $q_1(x)$). Counting the two non-zero coefficients of $f_3(u,v)$, we get the desired results.

6. Let AB be a diameter of k. Since $\sphericalangle DHA = \sphericalangle DCH = \alpha$, $\sphericalangle CHB = \sphericalangle CDH = \beta$, $\sphericalangle DMH = 2\alpha$, $\sphericalangle CMH = 2\beta$ and $\sphericalangle MDC = 90^o - \alpha - \beta$, the Sine theorem for $\triangle DMO$ gives



(1) $$MO = \frac{r\cos(\alpha+\beta)}{\cos(\alpha-\beta)},$$

where r is the radius of k_1 and $O = MH \cap CD$. On the other hand, if $MH \cap k = P$, then

$$\sphericalangle MAP = \frac{1}{2}\widehat{MP} = \frac{1}{2}(\widehat{DM} + \widehat{CP}) = \sphericalangle DOM = 90^o - \alpha + \beta.$$

Applying the Sine theorem for $\triangle APM$ we get that $2r = MP = 2R\cos(\alpha-\beta)$, where R is the radius of k. The Sine theorem for $\triangle DMC$ implies that

$$r = MC = 2R\cos(\alpha+\beta),$$

and hence $2\cos(\alpha+\beta) = \cos(\alpha-\beta)$. Now (1) shows that O is the midpoint of MH.

Conversely, it is easy to see that if O is the midpoint of MH, then H is the midpoint of MP, i.e. AB is a diameter of k.

Alternative solution. Let AB be a diameter of k and MH meets k for (the) second time at the point P. It is clear that $MH = HP = r$, where r is the radius of k_1. Consider the inversion with respect to the circle with center M and radius r. Then P is the image of the midpoint T of MH. The image of k is the line DC. Hence the image of P lies on DC, i.e. $T \in DC$.

Considering the same inversion also implies easily the converse statement.

7. We may assume the set A_1 has maximal cardinality. Denote $A_i \cap A_{i+1} = B_i, i = 1,2,\dots,n$. Since $A_n \supset B_{n-1} \cup B_n$, then

$$\begin{aligned} |A_n| &\geq |B_{n-1} \cup B_n| = |B_{n-1}| + |B_n| - |B_{n-1} \cap B_n| \\ &> \frac{n-2}{n-1}|A_n| + \frac{n-2}{n-1}|A_1| - |B_{n-1} \cap B_n|. \end{aligned}$$

Hence

$$|B_{n-1} \cap B_n| > \frac{n-2}{n-1}|A_1| - \frac{1}{n-1}|A_n| \geq \frac{n-3}{n-1}|A_1|,$$

i.e., $|A_{n-1} \cap A_n \cap A_1| > \dfrac{n-3}{n-1}|A_1|$. Further, if $C = A_{n-1} \cap A_n \cap A_1$, then $A_{n-1} \supset C \cup B_{n-2}$ and

$$\begin{aligned} |A_{n-1}| &\geq |B_{n-2} \cup C| = |B_{n-2}| + |C| - |B_{n-2} \cap C| \\ &> \frac{n-2}{n-1}|A_{n-1}| + \frac{n-3}{n-1}|A_1| - |B_{n-2} \cap C|. \end{aligned}$$

So $|B_{n-2} \cap C| > \dfrac{n-3}{n-1}|A_1| - \dfrac{1}{n-1}|A_{n-1}| \geq \dfrac{n-4}{n-1}|A_1|$, i.e.

$$|A_{n-2} \cap A_{n-1} \cap A_n \cap A_1| > \frac{n-4}{n-1}|A_1|.$$

We get by induction that

$$|A_{n-k} \cap A_{n-k+1} \cap \cdots \cap A_{n-1} \cap A_n \cap A_1| > \frac{n-k-2}{n-1}|A_1|$$

for $k = 1, 2, \ldots, n-2$. In particular, $|A_2 \cap A_3 \cap \cdots \cap A_{n-1} \cap A_n \cap A_1| > 0$.

8. The function $K(n)$ is increasing for $n \geq 3$, since $\displaystyle\sum_{i=1}^{n} \frac{1}{x_i} = 1$ implies that

$$\sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n+1} + \frac{1}{x_n(x_n+1)} = 1.$$

Thus it is enough to find $t \leq L(b)$ such that $K(t+2) + 2L(b) \geq d(b)$, where $d(b)$ is the number of the distinct positive integers that divide b. Let t be the minimal positive integer, for which the equation $\displaystyle\sum_{i=1}^{t} \frac{1}{x_i} = 1 - \frac{1}{b}$ has a solution. . Then $t \leq L(b)$. Fix now $t, b, x_1, \ldots, x_t$.

Note that the numbers of the solution of the equation $\dfrac{1}{y_1} + \dfrac{1}{y_2} = \dfrac{1}{b}$ such that b divides y_2 and $y_1 \leq y_2$, is equal to $d(b)$. Indeed, if $\dfrac{1}{b} = \dfrac{1}{y_1} + \dfrac{1}{kb}$ for some $k \geq 2$, then $y_1 = b + \dfrac{b}{k-1}$. So $k-1$ divides b and there are exactly $d(b)$ possibilities for k.

Hence $K(t+2)$ is not less than $d(b)$ minus the number of the cases, when $y_i = x_j$. This cases are at most $2L(b)$ which implies the desired inequality.

Team selection test for 45. IMO

1. We shall use the following well-known facts.

LEMA 1. *For any integer x and any positive integer t the number $t!$ divides $(x+1)(x+2)\dots(x+t)$.*

Proof. The statement is obvious for $x \in \{0, -1, -2, \dots, -t\}$. We have to prove it for $x > 0$. Let p be a prime divisor of $t!$. Then the power of p in the prime factorization of $t!$ is $\left[\frac{t}{p}\right] + \left[\frac{t}{p^2}\right] + \cdots$. This power does not exceed the power of p in the prime factorization of $(x+1)(x+2)\dots(x+t) = \frac{(x+t)!}{x!}$ since the last power equals $\left[\frac{x+t}{p}\right] + \left[\frac{x+t}{p^2}\right] + \cdots - \left[\frac{x}{p}\right] - \left[\frac{x}{p^2}\right] - \cdots = \left[\frac{x+t}{p}\right] - \left[\frac{x}{p}\right] + \left[\frac{x+t}{p^2}\right] - \left[\frac{x}{p^2}\right] - \cdots$ and $[a+b] \geq [a] + [b]$.

LEMMA 2. *For $g(x) \in \mathbb{R}[x]$ with $\deg g = n$ one has that $g(\mathbb{Z}) \subset \mathbb{Z}$ if and only if $g(x) = \sum_{i=0}^{n} b_i \binom{x}{i}$, where $b_0, b_1, \dots, b_n \in \mathbb{Z}$.*

Proof. Obviously, if g has the above form, then $g(\mathbb{Z}) \subset \mathbb{Z}$.

Conversely, let $g(\mathbb{Z}) \subset \mathbb{Z}$. Set $g(i) = \alpha_i \in \mathbb{Z}$, $i = 0, 1, \dots, n$ and apply the Lagrange interpolation formula with knots $0, 1, \dots, n$. Then

$$
\begin{aligned}
g(x) &= \sum_{i=1}^{n} \frac{x(x-1)\dots(x-i+1)(x-i-1)\dots(x-n)\alpha_i}{i(i-1)\dots(i-(i-1))(i-(i+1))\dots(i-n)} \\
&= \sum_{i=1}^{n} (-1)^{n+i} \frac{\alpha_i (n+1)\binom{x}{n+1}}{x-i}.
\end{aligned}
$$

It remains to use that any polynomial

$$
\frac{(n+1)\binom{x}{n+1}}{x-i} = \frac{x(x-1)\dots(x-i+1)(x-i+1)\dots(x-n)}{n!}
$$

can be written in the given form (compare the respective coefficients). The lemma is proved.

Let now m be a divisor of $n!$ and consider the polynomial $f(x) = (x+1)(x+2)\dots(x+n)$. It follows by lemma 1 that m divides $f(k)$ for any $k \in \mathbb{Z}$. Moreover, f is a monic polynomial (i.e, the leading coefficient of f equals 1) and hence $(a_0, a_1, \dots a_n, m) = 1$. So all divisors of $n!$ are solutions of the problem.

Assume that m does not divide $n!$ and m is a solution of the problem. Set $r = \frac{m}{(m, n!)}$. It is clear that $r > 1$ is an integer and $(r, n!) = 1$. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_n \neq 0$, be a polynomial with the desired properties. Then for $g(x) = \frac{f(x)}{r}$ one has that $g(\mathbb{Z}) \subset \mathbb{Z}$ and, by Lemma 2, $g(x) = \sum_{i=0}^{n} b_i \binom{x}{i}$, where $b_0, b_1, \dots, b_n \in \mathbb{Z}$.

Hence $f(x) = \sum_{i=0}^{n} mb_i \binom{x}{i}$ and using that $(r, i!) = 1$ for $i = 0, 1, \ldots, n$, we get that r divides all the coefficients of f, a contradiction.

2. It is easy to see that the remainder p modulo q is equal to $p - \left[\frac{p}{q}\right] q$. A direct verification shows that $p = 3$, 5, 7 and 13 are solutions of the problem. Assume that $p \geq 11$ is a solution and let q be a prime divisor of $p - 4$. If $q > 3$, the remainder under consideration is 4 – not square-free. So $q = 3$ and $p = 3^k + 4$ for some $k \in \mathbb{N}$. Analogously, the prime divisors of $p - 8$ are 5 or 7, and these of $p - 9$ are 2 or 7. Since 7 cannot be a divisor in both cases, we get that $p = 5^m + 8$ or $p = 2^n + 9$. Hence $3^k = 5^m + 4$ or $3^k = 2^n + 5$.

In the first case, $3^k \equiv 1 (\text{mod } 4)$, i.e. $k = 2k_1$, and so $(3^{k_1} - 2)(3^{k_1} + 2) = 5^m$ which gives $k_1 = 1$, $m = 1$, i.e. $k = 2$. In the second case one has that $n \geq 2$ and we conclude as above (using modulo 4 and 3) that $k = 2k_1$ and $n = 2n_1$ are even integers. Then $(3^{k_1} - 2^{n_1})(3^{k_1} + 2^{n_1}) = 5$ which implies $k_1 = n_1 = 1$, i.e. $k = n = 2$. Thus, in both cases, new solutions do not appear.

3. It it easy to see that if a triangle contains another triangle, then its inradius is greater than the inradius of the second one. So we may consider triangles with vertices on the boundary of the square. Moreover, we may assume that at least one vertex of the triangles is a vertex of the square and the other two vertices belong to the sides of the square containing no the first vertex. So we shall consider ΔOAB such that $O = (0, 0)$, $A = (a, 1)$ и $B = (1, b)$, $0 \leq a, b \leq 1$.

Consider also ΔOCD, where $C = (a+b, 1)$ and $D = (1, 0)$. Denote by S and P the area and perimeter of ΔOAB, respectively. Set $x = OA = \sqrt{1 + a^2}$, $y = AB = \sqrt{(1-a)^2 + (1-b)^2}$, $z = OB = \sqrt{1 + b^2}$, $u = OC = \sqrt{1 + (a+b)^2}$ and $v = CD = \sqrt{1 + (1 - a - b)^2}$. Note that $OD = 1$, $u \geq z \geq 1$, $x \geq 1$ and $v \geq 1$.

Comparing the perimeters of ΔOAB and ΔOCD gives

$$\begin{aligned}
(u + v + 1) - (x + y + z) &= \frac{u^2 - x^2}{u + x} + \frac{v^2 - y^2}{v + y} + \frac{1 - z^2}{1 + z} \\
&= \frac{2ab + b^2}{u + x} + \frac{2ab}{v + y} - \frac{b^2}{1 + z} \\
&\leq \frac{2ab + b^2}{1 + z} + \frac{2ab}{v + y} - \frac{b^2}{1 + z} \\
&= 2ab \left(\frac{1}{v + y} + \frac{1}{1 + z} \right) \leq 3ab \leq (u + v + 1)ab.
\end{aligned}$$

Hence

$$(u + v + 1)(1 - ab) \leq x + y + z \iff \frac{1}{u + v + 1} \geq \frac{1 - ab}{x + y + z} = \frac{2S}{P} = r.$$

On the other hand,

$$u+v+1=\sqrt{1+(a+b)^2}+\sqrt{1+(1-a-b)^2}+1\geq \min_{x\geq 1} F(x),$$

where $F(x)=\sqrt{1+x^2}+\sqrt{1+(1-x)^2}+1$. Since $\min\limits_{x\geq 1} F(x)=\sqrt{5}+1$, we get that $r\leq \dfrac{1}{\sqrt{5}+1}=\dfrac{\sqrt{5}-1}{4}$.

4. Let $x_1+x_2+\cdots+x_k=2004$, $x_1,x_2,\ldots,x_k\in\mathbb{N}$, $x_1<x_2<\cdots<x_k$ and the product $x_1x_2\ldots x_k$ is maximal. Assume that for some i,j, $1\leq i<j\leq k$ one has that $x_i\leq x_{i+1}-2$ and $x_j\leq x_{j+1}-2$. Then replacing x_i and x_j by x_i+1 and x_j-1, respectively (the sum is the same, i.e. 2004), we get a larger product since

$$(x_i+1)(x_j-1)=x_ix_j+x_j-x_i-1>x_ix_j,$$

a contradiction. Hence $x_1,x_2,\ldots,x_k$ are consecutive integers but at most one.

Let the numbers be $\{x_1,x_2,\ldots,x_k\}=\{x,x+1,\ldots,x+\ell,x+\ell+n,x+\ell+n+1,\ldots,x+k+n-2\}$ and $k=n+\ell-2$. If $n\geq 3$, we replace $x+\ell$ and $x+\ell+n$ by $x+\ell+1$ and $x+\ell+n-1$, respectively, and, as above, we get a larger product.

Let $n=1$ and the numbers be $x,x+1,\ldots,x+k-1$. If $x\geq 5$ we replace x by the numbers $x-2$ and 2. The sum remains 2004 and the product increases, since $2(x-2)>x$. If $1\leq x\leq 4$, a direct verification shows that either we have a larger product or the sum is not equal to 2004 (for $x=2$ and $x=3$).

It remains to consider the case $n=2$. Let the numbers be

$$x,x+1,\ldots,x+\ell,x+\ell+2,x+\ell+3,\ldots,x+k,\quad \ell\geq 0,k\geq \ell+2.$$

As above, we get a larger product for $x=1$ and $x\geq 4$.

If $x=2$, then $2+3+\cdots+(\ell+2)+(\ell+4)+\cdots+(k+2)=2004$ and hence $(k+2)(k+3)=2(2008+\ell)$. Since $0\leq \ell\leq k-2$, it follows that $4016\leq (k+2)(k+3)\leq 4012+2k$ and then $k=61$, $\ell=8$. So the numbers are $2,3,\ldots,10,12,13,\ldots,63$ with product $\dfrac{63!}{11}$.

For $x=3$ we get analogously $k=60$, $\ell=5$ and the numbers are $3,4,\ldots,8,10,11,\ldots,63$ with product $\dfrac{63!}{18}$ which is smaller than $\dfrac{63!}{11}$.

5. Let R, R_1, R_2 и R_3, O, O_1, O_2 и O_3 be the circumradii and circumcenters of $\triangle ABC$, $\triangle AHB$, $\triangle BHC$ and $\triangle CHA$, respectively. The Sine theorem implies that $R=R_1=R_2=R_3$. Then

$$\sin\sphericalangle C_1AH=\frac{C_1H}{2R_1}=\frac{A_1H}{2R_2}=\sin\sphericalangle A_1BH$$

and analogously

$$\sin\sphericalangle C_1AH=\sin\sphericalangle A_1BH=\sin\sphericalangle A_1CH=\sin\sphericalangle C_1BH=\sin\sphericalangle B_1AH.$$

Let $\sphericalangle C_1AH = \sphericalangle A_1BH = \sphericalangle B_1CH = \varphi$. Since $\sphericalangle AHB = \alpha+\beta = 180^\circ-\gamma$, then $\sphericalangle AC_1B = \gamma$, and hence $\sphericalangle AH_3B = 180^\circ - \gamma$. Analogously, $\sphericalangle BA_1C = \alpha$, $\sphericalangle BH_1C = 180^\circ - \alpha$, $\sphericalangle CB_1A = \beta$ и $\sphericalangle CH_2A = 180^\circ - \beta$. It follows that the points H_1, H_2 and H_3 belong to the circumcircle of $\triangle ABC$.

Now we shall use now vectors. Denote by $[\vec{a}]$ the image of the vector $\vec{a}$ under rotation through $360^\circ - 2\varphi$. We get from the proved above that $\sphericalangle C_1AH_3 = 90^\circ - \gamma = \sphericalangle CAH$. Hence $\sphericalangle CAH_3 = \varphi$ and then $\sphericalangle COH_3 = 2\varphi$. Analogously, $\sphericalangle AOH_1 = \sphericalangle BOH_2 = 2\varphi$.

Let S and T be the orthocenters of $\triangle A_1B_1C_1$ and $\triangle H_1H_2H_3$, respectively. We shall prove that $\overrightarrow{HS}=\overrightarrow{HT}$.

One has that

$$\begin{aligned}
\overrightarrow{HS} &= \overrightarrow{HA_1} + \overrightarrow{HB_1} + \overrightarrow{HC_1} = \overrightarrow{HO} + \overrightarrow{HA_1} + \overrightarrow{HO} + \overrightarrow{HB_1} + \overrightarrow{HO} + \overrightarrow{HC_1} \\
&= 3\,\overrightarrow{HO} + \overrightarrow{OO_1} + \overrightarrow{O_1A_1} + \overrightarrow{OO_2} + \overrightarrow{O_2B_1} + \overrightarrow{OO_3} + \overrightarrow{O_3C_1} \\
&= 3\,\overrightarrow{HO} + 2(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + \overrightarrow{O_1A_1} + \overrightarrow{O_2B_1} + \overrightarrow{O_3C_1} \\
&= \overrightarrow{HO} + [\overrightarrow{O_1H}] + [\overrightarrow{O_2H}] + [\overrightarrow{O_3H}] = \overrightarrow{HO} + [\overrightarrow{O_1O} + \overrightarrow{O_2O} + \overrightarrow{O_3O} + 3\,\overrightarrow{OH}] \\
&= \overrightarrow{HO} + [2(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + 3\,\overrightarrow{OH}] = \overrightarrow{HO} + [\overrightarrow{OH}], \\
\overrightarrow{HT} &= \overrightarrow{HO} + \overrightarrow{OT} = \overrightarrow{HO} + \overrightarrow{OH_1} + \overrightarrow{OH_2} + \overrightarrow{OH_3} \\
&= \overrightarrow{HO} + [\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}] = \overrightarrow{HO} + [\overrightarrow{OH}],
\end{aligned}$$

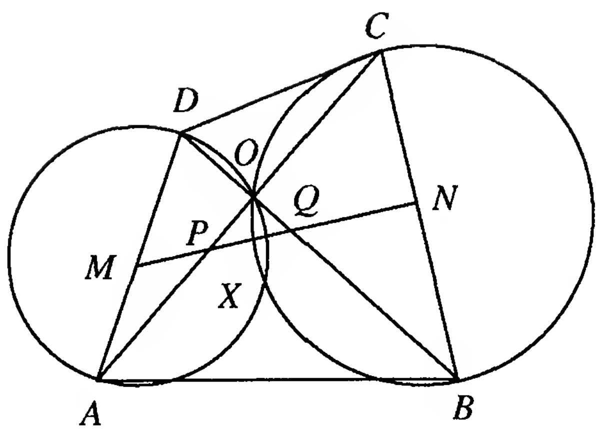
which means that $S \equiv T$.

The case $\sphericalangle C_1AH = \sphericalangle A_1CH = \sphericalangle B_1CH = \varphi$ can be considered in a similar way and we omit the details.

6. Consider a graph with vertices the lines of the table and edges that joint two vertices if the respective lines are different exactly in one position. Write on the edges the respective elements that are different.

Assume that the given statement is not true. Then the graph has n vertices and n edges. It is not difficult to prove by induction on n that there is a cycle in the graph. We may suppose that this cycle is $A_1A_2\dots A_k$ Starting from A_1, we remove x_1 to get A_2, then remove or add x_2 to get A_3, etc. When we turn back to A_1, we obtain a second copy of A_1 which does not contain x_1 (any edge contains no x_1), a contradiction.

7. Let $AC \cap BD = O$ and X be the second intersection point of the circumcircles of $\triangle AOB$ and BOC. Set $\sphericalangle XBO = \sphericalangle XCO = \alpha$ and $\sphericalangle XAO = \sphericalangle XDO = \beta$. Since $\triangle AXC \sim \triangle DXB$, then $\dfrac{XD}{XA} = \dfrac{BD}{AC}$. This and the condition of the problem implies that $\dfrac{AP}{AC} = 1 - \dfrac{BQ}{BD} = \dfrac{DQ}{BD}$ and hence $\dfrac{AP}{DQ} = \dfrac{AC}{BD} = \dfrac{XA}{XD}$. Then $\triangle APX \sim \triangle DQX$ and so $\sphericalangle APX = \sphericalangle DQX$. The last equality means that the points X, Q, O and P are concyclic and thus $\sphericalangle XQP = \sphericalangle XOP = \sphericalangle XDA$.



This implies that the points X, Q, D and M are concyclic, i.e. the circumcircle of $\triangle DMQ$ pases through X. Then $\sphericalangle XMN = \sphericalangle\beta$ and hence the points X, A, P and M are concyclic, i.e., the circumcircle of $\triangle AMP$ passes through X.

It follows in the same way that the circumcircle of $\triangle CNP$ passes through X and then analogously the circumcircle of $\triangle BNQ$ passes through X.

8. We call a graph, satisfying the given condition, n-purple. Let $f(n)$ be the smallest possible number of blue edges in an n-purple graph.

Suppose that $f(n) < n+5$ for $n \geq 5$. If any vertex of a n-purple graph G with $f(n)$ blue edges is a head of at least two blue edges, then the total number of the blue edges is at least $2n$. Since $2n \geq n+5 > f(n)$ for $n \geq 5$, we may find a vertex a of G that is the head of at most one blue edge. If a is not head of a blue edge, then there is vertex b of the graph $G \setminus \{a\}$ that is a head of a blue edge and hence $G \setminus \{a, b\}$ is a $(n-1)$-purple graph. If a and b are jointed by blue edge, then $G \setminus \{a, b\}$ is a $(n-1)$-purple graph. In both cases the obtained $(n-1)$-purple graph has at least one blue edge less than G and so $f(n) \geq f(n-1)+1$. Then $f(n-1) < n+4$, in particular, $f(4) < 9$.

Now we shall compute $f(4)$. It is well-known that there are 3-purple graphs. Then the above arguments show that any vertex of a 4-purple graph is a head of at least two blue edges. Hence $f(4) \geq 8$. If $f(4) = 8$, then any vertex is a head of exactly two blue edges. There are two such graphs containing no blue triangles but these graphs are not purple. If $f(4) = 9$, the two vertices are heads of three blue edges, and the remaining six vertices are heads of two blue edges. There are six such graphs containing no blue triangles but they are not purple. The following example shows that $f(4) = 10$ – a regular octagon with blue sides and two blue adjacent main diagonals, and the remaining diagonals are red.

So $f(n) \geq n+5$ for $n \geq 5$. It is easy to see that the graph with $2n$ vertices such that its blue edges form three disjoint cycles with lengths 5, 5 and $2n-10$ is n-purple. Hence $f(n) = n+5$ for $n \geq 5$.

9. Let the given numbers be $a_1, a_2, \ldots, a_{2n+1}$. Choose first 1 and then choose at any step (if it is possible) a number that is not a linear combination

with rational coefficients of the already chosen numbers. We may assume that the chosen numbers are $a_0 = 1, a_1, a_2, \dots, a_k$, $1 \le k \le 2n+1$. It is easy to see that any linear combination with rational coefficients of the given numbers can be uniquely presented linear combination of these numbers.

Let $a_i = \sum_{j=0}^{k} \alpha_{ij} a_j$, where $\alpha_{ij} \in \mathbb{Q}$, $1 \le i \le 2n+1$, $0 \le j \le k$. Then a sum of a_i's is a rational number if and only if the sum of the corresponding numbers $b_i = a_i - \alpha_{i0}$ vanishes. Since $b_1, b_2, \dots, b_{2n+1}$ are irrational numbers, they are non-zero. In particular, at least $n+1$ of them have the same sign and hence the corresponding a_i's have the desired property.

10. Using (d) for $y = 1$ and (c) we see that $f(zx, z) = z^k x$. Set $zx = y$. Then $f(y, z) = yz^{k-1}$ for $y \le z$. Conversely, if $y \le z$, then we set $y = zx$, $x \in [0, 1]$, and $f(y, z) = f(zx, z) = z^k x = yz^{k-1}$.

Now let $x \le y \le z$, $x, y, z \in (0, 1)$. Then using (a), we get that $f(xy^{k-1}, z) = f(x, yz^{k-1})$. Hence it follows from the above that

$$\{xy^{k-1}z^{k-1}, x^{k-1}y^{(k-1)^2}z\} \cap \{xy^{k-1}z^{(k-1)^2}, x^{k-1}yz^{k-1}\} \ne \phi.$$

This easily implies that either $k = 1$ and $f(x, y) = \min\{x, y\}$ or $k = 2$ and $f(x, y) = xy$. A direct verification shows that both functions are solutions for the respective k.

11. Set $a = \dfrac{9x^2}{x^2+y^2+z^2}$, $b = \dfrac{9y^2}{x^2+y^2+z^2}$, $c = \dfrac{9z^2}{x^2+y^2+z^2}$, where $x, y, z > 0$ and $x + y + z = 1$. We have to prove that

$$x^2 + y^2 + z^2 \ge 9(x^2y^2 + y^2z^2 + z^2x^2).$$

Since $a \ge 1$, then

$$9x^2 = a(x^2 + y^2 + z^2) \ge x^2 + y^2 + z^2 \ge \frac{(x+y+z)^2}{3} = \frac{1}{3},$$

т.е. $x \ge \dfrac{1}{3\sqrt{3}}$. Analogously $y \ge \dfrac{1}{3\sqrt{3}}$ и $z \ge \dfrac{1}{3\sqrt{3}}$.

We may assume that $x \ge y \ge z$. Then

$$\frac{1}{3\sqrt{3}} \le z \le \frac{1}{3}, \quad \frac{1}{3} \le \frac{x+y}{2} = \frac{1-z}{2} \le \frac{3\sqrt{3}-1}{6\sqrt{3}}.$$

Let $f(x, y, z) = x^2 + y^2 + z^2 - 9(x^2y^2 + y^2z^2 + z^2x^2)$. Then

$$f(x,y,z) - f\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right) = \frac{(x-y)^2}{2}\left[1 - 9z^2 + \frac{9}{8}((x+y)^2 + 4xy)\right] \ge 0.$$

It is enough to prove that $f(t, t, 1-2t) \ge 0$, where $t = \dfrac{x+y}{2}$. One has that

$$\begin{aligned} f(t, t, 1-2t) &= 2t^2 + (1-2t)^2 - 9(t^4 + 2t^2(1-2t)^2) \\ &= \frac{(p-1)^2}{3}(3 + 2p - 3p^2) \ge 0, \end{aligned}$$

where $p = 3t$ $(1 \le p \le \dfrac{3\sqrt{3}-1}{2\sqrt{3}})$, i.e., $3 + 2p - 3p^2 > 0 \iff \dfrac{1-\sqrt{10}}{3} < p < \dfrac{1+\sqrt{10}}{3}$. It remains to check that $\dfrac{3\sqrt{3}-1}{2\sqrt{3}} < \dfrac{1+\sqrt{10}}{3}$.

In fact, we have proved the inequality under the weaker conditions $a+b+c = 9$ and $a, b, c \ge \dfrac{89-28\sqrt{10}}{3}$.

12. We put $m+n-1$ white pieces by $m+n-1$ movies consecutively in the cells $(1,1), (2,1), \dots, (m,1), (1,2), (1,3), \dots, (1,n)$.

We shall call a closed chain of cells such that at least two numbers of rows and columns change alternatively a zig-zag cycle.

Note that there is at least one black piece in the cells of a zig-zag cycle. Indeed, assume the contrary and consider the last white piece. Its neighbors (in the zig-zag cycle) are white pieces that have been put before it. Then the last piece must be black, a contradiction.

Assume now that the game is over and there are more than $m+n-1$ pieces. Remove a row or a column with at most one white piece, repeat the same operation, etc. Since such a removing can be done at most $m+n-1$ times, any row and column in the new table will contain at least two white pieces. This new table, and hence the given table, has a zig-zag cycle, a contradiction to the proved above.

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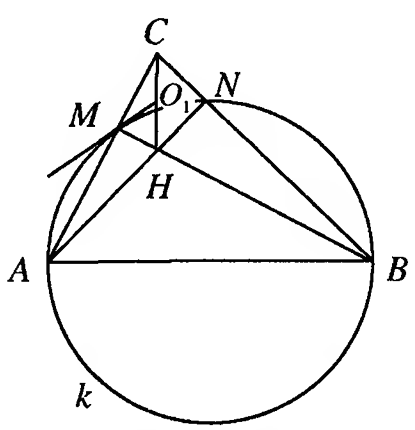
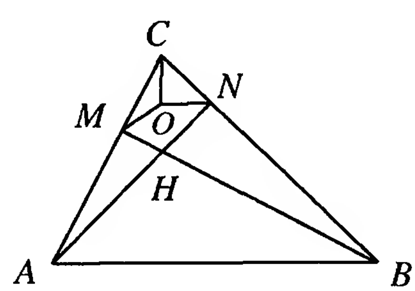
9.1. For $a \neq 0$, $a \neq 1$, the given equality is equivalent to

$$\begin{aligned} a(x_1x_2 + x_3x_4) &= x_1x_2x_3x_4(x_1 + x_2 + x_3 + x_4) \\ \iff 2a - 1 = (a-1)(a+5) &\iff a^2 + 2a - 4 = 0 \\ &\iff a_{1,2} = -1 \pm \sqrt{5}. \end{aligned}$$

It is easy to check that for these values of a both equations have real roots. The case $a = 0$ is excluded by the condition and $a = 1$ implies $x_4 = 0$, whence $x_1 = 0$, which is a contradiction.

9.2. If AB is a diameter of k, then AN and BM are altitudes of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$ and let the tangent line to k at M meet the altitude CH at O_1. Then $\sphericalangle CMO_1 = \sphericalangle ABM = \frac{\overset{\frown}{AM}}{2}$ and $\sphericalangle ABM = \sphericalangle ACH$. Thus $\sphericalangle CMO_1 = \sphericalangle MCO_1$, i.e. $CO_1 = MO_1$. On the other hand $\sphericalangle O_1HM = 90^\circ - \sphericalangle O_1CM = 90^\circ - \sphericalangle CMO_1 = \sphericalangle O_1MH$, i.e. $O_1M = O_1H$. Therefore O_1 is the midpoint of CH. It can be seen analogously that the tangent line to k at N passes through O_1, i.e. $O \equiv O_1$ and $OM = ON = OC = CH/2$.





Let O be the circumcenter of $\triangle CMN$. Then $\sphericalangle CMO = \sphericalangle MCO = \sphericalangle ABM$ and $\sphericalangle CNO = \sphericalangle NCO = \sphericalangle BAN$. Hence

$$\sphericalangle ACB = \sphericalangle MCO + \sphericalangle NCO = \sphericalangle ABM + \sphericalangle BAN.$$

Therefore

$$\begin{aligned} 2\sphericalangle ANB &= \sphericalangle ANB + \sphericalangle AMB \\ &= 180^\circ - \sphericalangle ABC - \sphericalangle BAN + 180^\circ - \sphericalangle BAC - \sphericalangle ABM \\ &= 360^\circ - (\sphericalangle ABC + \sphericalangle ACB + \sphericalangle BAC) = 180^\circ. \end{aligned}$$

Hence $\sphericalangle ANB = 90^\circ$ and AB is a diameter of k.

9.3. The number $m - n$ has at most three different positive divisors if and only if $m - n = p^k$, where p is a prime and $k \in \{0, 1, 2\}$. If $k = 0$ then $m = n + 1$

and $n(n+1)$ is a perfect square, which is impossible. Let $m-n=p^k$, $mn=t^2$, where $k \in \{1,2\}$ and t is a positive integer. Then

$$n(n+p^k)=t^2 \iff (2n+p^k-2t)(2n+p^k+2t)=p^{2k}.$$

Therefore $2n+p^k-2t=p^s$ and $2n+p^k+2t=p^r$, where r and s are integers, such that $0 \le s < r \le 2k$ and $r+s=2k$.

For $k=1$ we have the unique possibility $2n+p-2t=1$, $2n+p+2t=p^2$. Hence

$$n=\frac{(p-1)^2}{4}, \quad m=n+p=\frac{(p+1)^2}{4}.$$

Taking into account that $1000 \le m < 2005$ we obtain the solutions $m=1764$, 1600, 1369, 1296 and 1156 (for $p=83$, 79, 73, 71 and 67, respectively).

For $k=2$ we have $(r,s)=(4,0)$ or $(r,s)=(3,1)$. In the first case we get

$$n=\frac{(p^2-1)^2}{4}, \quad m=n+p^2=\frac{(p^2+1)^2}{4}.$$

Now the inequalities $1000 \le m < 2005$ imply that $p=8$ which is not a prime. In the second case we have $m=p(p+1)^2/4$, which gives the solutions $m=1900$ and 1377 (for $p=19$ and 17, respectively).

9.4. Yana has at least one card, say $k \neq 1$. If Ivo has 1, then the product $1.k=k$ does not belong to Yana, a contradiction. Therefore Yana has 1.

If 12 is in Ivo, then the sum $13=1+12$ belongs to Yana, a contradiction. Therefore 12 belongs to Yana. Since the sum $13=6+7$ is in Ivo, both cards 6 and 7 belong to one and the same person. They are not in Ivo since otherwise the sum $1+6=7$ is in Yana. Using similar arguments we conclude that all cards $1,2,\ldots,12$ belong to Yana. Further, all cards $13k$, $k=1,\ldots,7$, are in Ivo, and all the others belong to Yana. Therefore Yana has $100-7=93$ cards.

10.1. a) We consider two cases. If $x \in (-\infty,2] \cup [3,\infty)$, then the inequality becomes $x^2-6x+6 \le 0$, whence $x \in [3-\sqrt{3}, 3+\sqrt{3}]$. Therefore the solutions of the inequality are $x \in [3-\sqrt{3},2] \cup [3,3+\sqrt{3}]$.

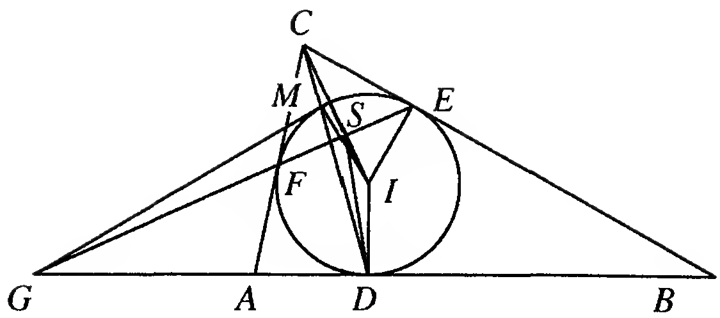
If $x \in (2,3)$, then the inequality becomes $x^2-4x+6 \ge 0$, which is satisfied for every $x \in (2,3)$. Thus $x \in [3-\sqrt{3},3+\sqrt{3}]$.

b) If the inequality $|x^2-5x+6| \le x+a$ has an integral solution x then $x \in (-\infty,2] \cup [3,\infty)$ and therefore

$$x^2-6x+6-a \le 0.$$

This inequality has a solution if and only if $a \ge -3$ and in this case we have that $x \in [3-\sqrt{a+3}, 3+\sqrt{a+3}]$. This interval contains the number 3 and is symmetric with respect to 3. Therefore it contains exactly three integers if and only if $1 \le \sqrt{a+3} < 2$, i.e. $a \in [-2,1)$.

10.2. a) From the right $\triangle CEI$ we have $EI^2 = SI.CI = DI^2$. Then we get $\dfrac{DI}{SI}=\dfrac{CI}{DI}$ and therefore $\triangle CDI \sim \triangle DSI$.



b) The quadrilateral $DIMG$ is cyclic. Since a) implies that $\sphericalangle ISD = \sphericalangle IDC = \sphericalangle IMD$ we conclude that S lies on the circumcircle of $DIMG$. It is now obvious that $\sphericalangle GSI = \sphericalangle GMI = 90°$.

Remark. It easy to see that b) implies that the points E, F and G are colinear.

10.3. We set $x = u - 1$, $y = v + 1$ and obtain the equation

$$z^2 + 1 = (u^2 - 1)(v^2 - 1).$$

It is easy to see that u, v and z must be even. Hence, if $|u| > 1$, then $u^2 - 1$ has a prime divisor p such that $p \equiv 3 \pmod 4$. Therefore $z^2 + 1 \equiv 0 \pmod p$, which is impossible (it is well known that if p is a prime such that $p \equiv 3 \pmod 4$ and p divides $x^2 + y^2$ then p divides both x and y). Thus $u = 0$.

Analogously we get $v = 0$ and therefore $z = 0$. Hence $x = -1, y = 1, z = 0$.

10.4. We first prove that for any odd $n \geq 3$ there are $n \times n$ tables that are not "good". Consider an arbitrary $n \times n$ table and denote by P_i, $i = 1, 2, \ldots, n$, the product of the numbers in the i-th row at the second last step. Then $P_1P_3 = P_2P_4 = \cdots = P_{n-1}P_1 = P_nP_2 = 1$ and since n is odd, it follows that $P_1 = P_2 = \cdots = P_n$. This argument shows that the row products in the initial table must be equal. Therefore any table which has not this property is not "good".

We now consider a table of order $n = 2^k m$, where m is an odd number and $k \geq 1$. After the first two steps the number in the position (i, j) becomes equal to the product of the numbers in the positions $(i-2, j)$, $(i, j-2)$, $(i, j+2)$ and $(i+2, j)$. Therefore the resulting table after every even step can be obtained by applying the operation on the following four tables of order $2^{k-1}m$:

- the table of all (i, j) with $i \equiv j \equiv 0 \pmod 2$;
- the table of all (i, j) with $i \equiv 0 \pmod 2$, $j \equiv 1 \pmod 2$;
- the table of all (i, j) with $i \equiv 1 \pmod 2$, $j \equiv 0 \pmod 2$;
- the table of all (i, j) with $i \equiv j \equiv 1 \pmod 2$.

Now it follows by induction that the number $n = 2^k m$ has the required property if and only if the number $2^{k-1}m$ does. It is also easy to see that every table of order 2 is "good".

Therefore the required n are $n = 2^k$, where k is a positive integer.

11.1. a) Using the formulas for the sums of arithmetic and geometric progressions we obtain the equality

$$\frac{n[2m+2(n-1)]}{2}=n(2^m-1),$$

whence $m+n=2^m$.

b) It follows that $4n=m+44$. Using a), we obtain $2^{m+2}=44+5m$. It is easy to see that $m=4$ is a solution. If $m<4$ then $2^{m+2}\le 2^5<44+5m$. If $m>4$ then it follows by induction that $2^{m+2}>44+5m$. Therefore $m=4$ and $n=12$.

11.2. The equation is equivalent to $x\in(1,2)$ and $ax+1=(x-1)(2-x)$, which can be written as $x^2+(a-3)x+3=0$. Therefore we have to find the values of a, such that the equation

$$f(x)=x^2+(a-3)x+3=0$$

has exactly one root in the interval $(1,2)$. This is possible exactly in the following four cases:

Case 1. $f(1)f(2)<0$, which is equivalent to $a\in\left(-1,-\frac{1}{2}\right)$.

Case 2. $f(1)=0$, i.e. $a=-1$. Then $x_1=1$ and $x_2=3$, which shows that $a=-1$ is not a solution.

Case 3. $f(2)=0$, i.e. $a=-\frac{1}{2}$. Then $x_1=2$ and $x_2=\frac{3}{2}$, which shows that $a=-\frac{1}{2}$ is a solution.

Case 4. $D=0$, i.e. $(a-3)^2-12=a^2-6a-3=0$, whence $a=3\pm2\sqrt{3}$. For $a=3+2\sqrt{3}$ we have $x_1=x_2=-\sqrt{3}$, i.e. $a=3+2\sqrt{3}$ is not a solution. For $a=3-2\sqrt{3}$ we get $x_1=x_2=\sqrt{3}\in(1,2)$, i.e. $a=3-2\sqrt{3}$ is a solution.

Finally, $a\in\left(-1,-\frac{1}{2}\right]\cup\{3-2\sqrt{3}\}$.

11.3. Let $K=CO\cap AB$ and $\measuredangle AKC=\varphi$. Denote by M and N the intersection points of QO with CB and CA, respectively. The Sine theorem for $\triangle APQ$ gives

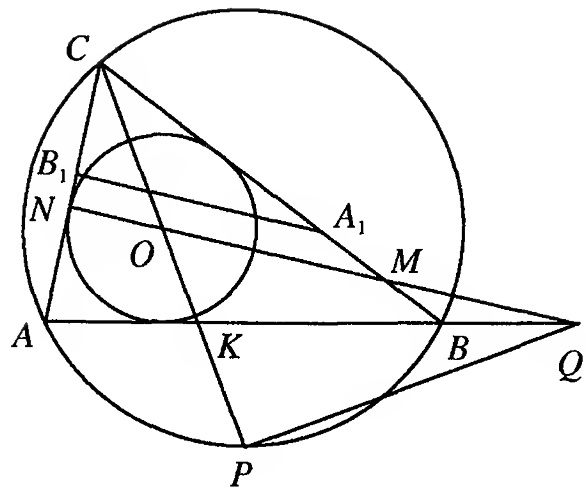
$$\frac{AQ}{PQ}=\frac{\sin(90^\circ+\beta)}{\sin\frac{\gamma}{2}}=\frac{\cos\beta}{\sin\frac{\gamma}{2}}.$$

From the right $\triangle KPQ$ we find

$$\frac{KQ}{PQ}=\frac{1}{\sin\varphi}.$$

Hence

(1) $$\frac{AQ}{QK}=\frac{\cos\beta\sin\varphi}{\sin\frac{\gamma}{2}}.$$



It follows from $\triangle AKC$ (AO is the bisector of $\measuredangle KAC$) that

$$(2) \qquad \frac{KO}{OC} = \frac{AK}{AC} = \frac{\sin\frac{\gamma}{2}}{\sin\varphi}.$$

On the other hand applying the Menelaus theorem for $\triangle AKC$ and the line OQ we get

$$(3) \qquad \frac{AQ}{QK} \cdot \frac{KO}{OC} \cdot \frac{CN}{NA} = 1.$$

Now plugging (1) and (2) in (3) we obtain $\dfrac{CN}{NA} = \dfrac{1}{\cos\beta}$. Hence it follows that $\dfrac{CN}{CA} = \dfrac{1}{1+\cos\beta}$, i.e. $CN = \dfrac{2R\sin\beta}{2\cos^2\frac{\beta}{2}} = 2R\tan\dfrac{\beta}{2}$.

Similarly, $CM = 2R\tan\dfrac{\alpha}{2}$. Therefore

$$(4) \qquad \frac{CN}{CM} = \frac{\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}}.$$

It is well known that $CB_1 = p - a = r\cot\frac{\alpha}{2}$ and $CA_1 = p - b = r\cot\frac{\beta}{2}$. Hence using (4) we get

$$\frac{CB_1}{CA_1} = \frac{\cot\frac{\alpha}{2}}{\cot\frac{\beta}{2}} = \frac{\operatorname{tg}\frac{\beta}{2}}{\operatorname{tg}\frac{\alpha}{2}} = \frac{CN}{CM}.$$

Therefore $A_1B_1 \parallel MN$.

11.4. Note that a chess player could not have more than one draw. Indeed, if A had draws with B and C, then the condition for A and B implies that B defeated C and the same condition for A and C implies that C defeated B, a contradiction.

Let $A_1, A_2, \dots, A_k$ be the longest sequence such that each chess player has defeated the next one, i.e. A_i has defeated A_{i+1} for $i = 1, 2, \dots k-1$. If $k = 2005$, then we have the required sequence. Assume that $k < 2005$ and consider a chess player B who is not amongst $A_1, A_2, \dots, A_k$.

If B has defeated A_1 then the sequence $B, A_1, A_2, \dots, A_k$ of length $k+1$ has the above property, which is impossible.

If A_1 has defeated B, then B and A_2 had not draw because A_1 has defeated both. If B has defeated A_2 then the sequence $A_1, B, A_2, \dots, A_k$ of length $k+1$ has the above property, a contradiction. Therefore A_2 has defeated B. We see analogously that all players $A_3, A_4, \dots, A_k$ have defeated B. Then we obtain again a contradiction by considering the sequence $A_1, A_2, \dots, A_k, B$ of length $k+1$.

The above argument shows that outside the sequence $A_1, A_2, \dots, A_k$ there is only one chess player B, and A_1 and B made a draw. Then this is the only draw of B. If A_2 has defeated B, then we obtain as above that B has lost from A_i for $i = 3, 4, \dots, k$ and we have again sequence $A_1, A_2, \dots, A_k, B$ of length $k+1$. Therefore A_2 has lost from B and the same holds for A_i, $i = 3, \dots, k$.

On the other hand, there is at least one more draw, for example between A_i and A_j. But A_i and A_j have lost from B, which is a contradiction.

12.1. a) Since $a_{n+1} + b_{n+1} = 2b_n - a_n + 2a_n - b_n = a_n + b_n$, we have

$$a_{n+1} = 2(a_n + b_n) - 3a_n = 2(a_1 + b_1) - 3a_n.$$

b) Using a), we obtain

$$a_{n+1} - \frac{a_1 + b_1}{2} = -3\left(a_n - \frac{a_1 + b_1}{2}\right),$$

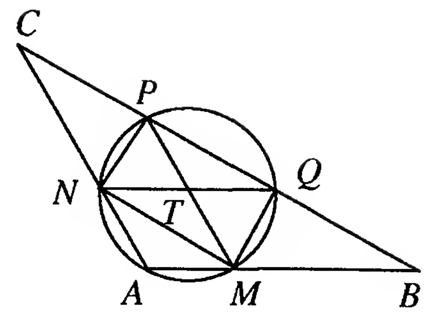
whence

$$a_{n+1} - \frac{a_1 + b_1}{2} = (-3)^n\left(a_1 - \frac{a_1 + b_1}{2}\right).$$

Since $\lim_{n\to\infty} 3^n = +\infty$, it follows that if $a_1 > b_1$, then $\lim_{n\to\infty} a_{2n} = -\infty$, a contradiction. Analogously, we see that it is not possible to have $a_1 < b_1$. Therefore $a_1 = b_1$.

12.2. Set $BC = a$, $CA = b$ and $AB = c$. Then $BA.BM = BP.BQ$, and

$$\frac{BM}{c} = \frac{BP}{a}.$$



Hence $BQ = \frac{c^2}{a}$ and analogously $CP = \frac{b^2}{a}$. Then $BP = \frac{a^2 - b^2}{a}$, $CQ = \frac{a^2 - c^2}{a}$ and the condition $\frac{BP}{CQ} = \frac{AB}{AC}$ becomes

$$b(a^2 - b^2) = c(a^2 - c^2), \text{ i.e. } (b - c)(a^2 - b^2 - c^2 - bc) = 0.$$

Since $b \neq c$, we get $a^2 - b^2 - c^2 - bc = 0$ and the Cosine theorem gives $\cos \measuredangle BAC = -\frac{1}{2}$. Therefore $\measuredangle BAC = 120°$.

Second solution. Since the quadrilateral $AMPN$ is a cyclic trapezoid, it follows that $AM = NP$. Also, if $T = MP \cap NQ$, then $AMTN$ is a parallelogram and $AM = NT$. Then $NP = NT$ and in the same way $MQ = MT$. Hence TPN and TQM are similar isosceles triangles and we have

$$\text{(1)} \qquad \frac{TP}{TQ} = \frac{TN}{TM}.$$

Using the Sine theorem, we obtain $\frac{MP}{\sin\beta} = \frac{BP}{\sin\alpha}$, $\frac{NQ}{\sin\gamma} = \frac{CQ}{\sin\alpha}$ and since $\frac{BP}{CQ} = \frac{AB}{AC} = \frac{\sin\gamma}{\sin\beta}$, we conclude that $MP = NQ$. From here and (1) it follows that

$$TM + TQ\frac{TN}{TM} = TN + TQ \iff (TM - TN)(TM - TQ) = 0.$$

Assume that $TM = TN$. We have $MQ = NP$ and $NA = MA$. The first of these identities shows that $MN || PQ$ and by the second one we obtain $AC = AB$, a contradiction.

Therefore $TM = TQ$, i.e. $\triangle MTQ$ is equilateral. Hence

$$\measuredangle BAC = \measuredangle MTN = 120°.$$

12.3. Set $t = \sin x$ and $g(t) = \frac{t^2 - a}{t^3 - (a^2 + 2)t + 2}$.

If the nominator and the denominator of g have a common root then $a \geq 0$ and $t = \pm\sqrt{a}$. If $t = -\sqrt{a}$ we obtain $\sqrt{a}(a^2 - a + 2) = -2$, which is impossible since $a^2 - a + 2 > 0$ for every a. If $t = \sqrt{a}$ we obtain $\sqrt{a}(a(a-1) + 2) = 2$ and it is easy to see that $a = 1$ is the only solution of this equation (for $a \in [0, 1)$ the left hand side is less than 2, and for $a > 1$ it is greater than 2). For $a = 1$ we have

$$g(t) = \frac{t^2 - 1}{t^3 - 3t + 2} = \frac{t + 1}{(t-1)(t + 2)} \leq 0$$

for every $t \in [-1, 1)$ which implies that this value of a is not a solution.

We now look for $a \neq 1$ such that the equation $g(t) = c$, i.e.

$$h(t) = c(t^3 - (a^2 + 2)t + 2) + a - t^2 = 0,$$

has a solution in the interval $[-1, 1]$ for every $c \in \left[\frac{1}{2}, 2\right]$. For $c \geq \frac{1}{2}$ we have

$$h(-1) = ca^2 + a + 3c - 1 \geq \frac{a^2}{2} + a + \frac{1}{2} = \frac{(a+1)^2}{2},$$

whence $h(-1) \geq 0$ for every a. Also, $h'(t) = 3ct^2 - 2t - c(a^2 + 2)$ and therefore $h'(1) = c - ca^2 - 2 \leq 0$ for $c \in \left[\frac{1}{2}, 2\right]$ and every a. Hence the equation $h'(t) = 0$ has real roots t_1 and t_2 such that $t_1 \leq 1 < t_2$.

Thus the function $h(t)$ is decreasing in the interval $[t_1, t_2] \ni 1$ and increasing in the interval $(-\infty, t_1]$. Since $h(-1) \geq 0$, it is easy to see that $h(t) = 0$ has a solution in the interval $[-1, 1]$ for every $c \in \left[\frac{1}{2}, 2\right]$ if and only if

$$h(1) = (a - 1)(1 - c(1 + a)) \leq 0$$

for every such c. For $a > 1$ this inequality is satisfied since $1 - \frac{1}{2}(1 + a) < 0$, and for $a < 1$ it is equivalent to $1 - 2(1 + a) \geq 0$, i.e. $a \leq -\frac{1}{2}$.

Finally, the required values of a are $a \in \left(-\infty, -\frac{1}{2}\right] \cup (1, +\infty)$.

Second solution. Using the same reasoning as above we reduce the problem to finding of those a such that $\left[\frac{1}{2}, 2\right] \subset g([-1, 1])$.

Set $h(t) = t^3 - (a^2 + 2)t + 2$. Then $h'(t) = 3t^2 - (a^2 + 2)$ and for $a^2 \geq 1$ it follows that $h'(t) < 0$ for $t \in (-1, 1)$. Therefore $h(t)$ is a strictly decreasing continuous function in the interval $(-1, 1]$. Since

$$h(1) = 1 - a^2 \leq 0 < a^2 + 3 = h(-1)$$

if follows that $h(t)$ has a unique zero $t_0 \in (-1, 1]$.

We now consider several cases.

Case 1. Let $a > 1$. Then $t_0 \neq 1$, $\lim\limits_{t \to t_0 + 0} g(t) = +\infty$ and $g(1) = \frac{1 - a}{1 - a^2} = \frac{1}{1 + a} < \frac{1}{2}$. Since the function $g(t)$ is decreasing and continuous in $(t_0, 1]$, it follows that

$$\left[\frac{1}{2}, 2\right] \subset \left[\frac{1}{2}, +\infty\right) \subset g((t_0, 1]) \subset g([-1, 1]).$$

Case 2. Let $a \le -1$. Now $\lim\limits_{t \to t_0 - 0} g(t) = +\infty$ and $g(-1) = \dfrac{1-a}{a^2+3} \le \dfrac{1}{2}$ (the last inequality is equivalent to $(a+1)^2 \ge 0$). Then

$$\left[\frac{1}{2}, 2\right] \subset \left[\frac{1}{2}, +\infty\right) \subset g([-1, t_0)) \subset g([-1, 1]).$$

Case 3. Let $a = 1$. Then we see as in the first solution that $g(t) \le 0$ for $t \in [-1, 1)$.

Case 4. Let $a \in (-1, 1)$. We first check that $h(t) > 0$ for $t \in [-1, 1]$, which implies that $g(t)$ is a continuous function in that interval.

Indeed, if $t \in [-1, 0]$, then $h(t) \ge t^3 + 2 > 0$ and if $t \in (0, 1]$, then $h(t) > t^3 - 3t + 2 = (t-1)^2(t+2) \ge 0$.

Case 4.1. If $a \in \left(-1, -\dfrac{1}{2}\right]$, then $g(-1) = \dfrac{1-a}{a^2+3} < \dfrac{1}{2}$, $g(1) = \dfrac{1}{1+a} \ge 2$ and therefore $\left[\dfrac{1}{2}, 2\right] \subset g([-1, 1])$.

Case 4.2. If $a \in \left(-\dfrac{1}{2}, 1\right)$, we shall show that $g(t) < 2$ for every $t \in [-1, 1]$. Since $h(t) > 0$ in this interval, we have to check that

$$t^2 < 2(t^3 - (a^2+2)t + 2),$$

i.e.

(1) $$m(t) = 2t^3 - t^2 - 2(a^2+2)t + a + 4 > 0, \ \ t \in [-1, 1].$$

We have

$$m'(t) = 6t^2 - 2t - 2(a^2+2) \le 6t^2 - 2t - 4 = (6t+4)(t-1)$$

and therefore $m'(t) \le 0$ in the interval $\left[-\dfrac{2}{3}, 1\right]$. Hence $m(t)$ is a decreasing function in this interval which implies that

$$m(t) \ge m(1) = -2a^2 + a + 1 = (1+2a)(1-a) > 0$$

for any $t \in \left[-\dfrac{2}{3}, 1\right]$. On the other hand, we have

$$m(t) \ge 2t^3 - t^2 + a + 4 \ge -2 - 1 - \frac{1}{2} + 4 > 0$$

for $t \in [-1, 0]$, which completes the proof of (1).

The cases considered above cover all possibilities for a and we conclude that the answer is $a \in \left(-\infty, -\dfrac{1}{2}\right] \cup (1, +\infty)$.

12.4. Using the standard notation for the elements of $\triangle ABC$ we have $b^2 = l_a^2 = bc - \dfrac{a^2bc}{(b+c)^2}$. Hence

$$a^2c = (c-b)(c+b)^2. \tag{1}$$

Let $\dfrac{a}{c} = \dfrac{m}{n}$, $(m,n)=1$, and $\dfrac{b}{c} = \dfrac{r}{s}$, $(r,s)=1$. Then (1) implies that

$$\frac{m^2}{n^2} = \frac{(s-r)(s+r)^2}{s^3}.$$

Since both sides are irreducible fractions, we obtain $m^2 = (s-r)(s+r)^2$ and $n^2 = s^3$. The first equality shows that $s-r$ is a perfect square and the second one implies that s is a perfect square. Set $s = t^2$ and $s - r = k^2$. Then $r = t^2 - k^2$, $m = k(2t^2 - k^2)$ and $n = t^3$.

We now set $a = mx$, $c = nx$, $b = ry$ and $c = sy$. Then $nx = sy$, i.e. $y = tx$. Therefore $a = xk(2t^2 - k^2)$, $b = xt(t^2 - k^2)$ and $c = xt^3$, where $t > k$ and $(t,k) = 1$. Moreover, one can easily check that these a, b and c satisfy the triangle inequality.

Now the condition $a+b+c = 10p$ becomes $x(k+t)(2t^2-k^2) = 10p$. Note that $(k+t, 2t^2-k^2) = 1$. Then it is easy to see that we have only the following possibilities:

$$\left|\begin{array}{l} x=1 \\ k+t=5 \\ 2t^2-k^2=2p \end{array}\right., \quad \left|\begin{array}{l} x=1 \\ k+t=10 \\ 2t^2-k^2=p \end{array}\right., \quad \left|\begin{array}{l} x=2 \\ k+t=5 \\ 2t^2-k^2=p \end{array}\right..$$

A direct verification shows that $(x,k,t) = (1,2,3)$, $(x,k,t) = (1,3,7)$ and $(x,k,t) = (2,1,4)$. Therefore $(a,b,c) = (28,15,27)$, $(a,b,c) = (267,280,343)$ and $(a,b,c) = (62,120,128)$.

8.1. Since both sides of the equation are non-negative, it is equivalent to

$$\left(\left|x-\frac{5}{2}\right|-\frac{3}{2}\right)^2=(x^2-5x+4)^2.$$

Hence

$$\left(\left|x-\frac{5}{2}\right|-\frac{3}{2}-x^2+5x-4\right)\left(\left|x-\frac{5}{2}\right|-\frac{3}{2}+x^2-5x+4\right)=0$$

and we have to solve the equations

$$\left|x-\frac{5}{2}\right|-x^2+5x-\frac{11}{2}=0$$

and

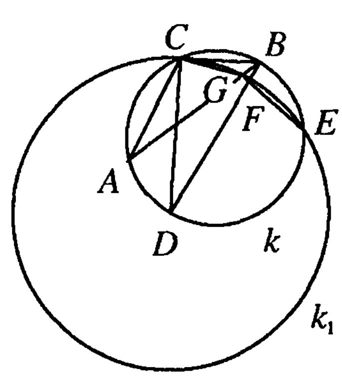
$$\left|x-\frac{5}{2}\right|+x^2-5x+\frac{5}{2}=0.$$

For $x\le\frac{5}{2}$ we have respectively that $\frac{5}{2}-x-x^2+5x-\frac{11}{2}=0 \iff x^2-4x+3=0$, i.e. $x=1$ or $x=3$ and $\frac{5}{2}-x+x^2-5x+\frac{5}{2}=0 \iff x^2-6x+5=0$, i.e. $x=1$ or $x=5$. Therefore $x=1$ is the only solution of the given equation in this case.

Analogously, for $x>\frac{5}{2}$ we have $x-\frac{5}{2}-x^2+5x-\frac{11}{2}=0 \iff x^2-6x+8=0$, i.e. $x=2$ or $x=4$ and $x-\frac{5}{2}+x^2-5x+\frac{5}{2}=0 \iff x^2-4x=0$, i.e. $x=0$ or $x=4$. Therefore $x=4$ is the only solution of the given equation in this case.

Hence the given equation has two solutions $x_1=1$ and $x_2=4$.

8.2. Since $\sphericalangle ACB>90°$, the points G and E lie on different sides to the diameter BD. If $\sphericalangle DCG=\sphericalangle EFD$, then $\sphericalangle DCG+\sphericalangle DFG=180°$, i.e. it is enough to prove that the quadrilateral $CDFG$ is cyclic. On the other hand, $CE\perp BD$ and D is the center of k_1. Therefore $\sphericalangle CDF=\frac{1}{2}\overset{\frown}{CGE}_{k_1}$ (as a central angle), $\sphericalangle CGF=\sphericalangle CGE$ is inscribed in k_1 and $\sphericalangle CGF=\frac{1}{2}(360°-\overset{\frown}{CGE}_{k_1})$.



Then $\sphericalangle CDF+\sphericalangle CGF=180°$, i.e. the quadrilateral $CDFG$ is cyclic, whence $\sphericalangle DCG=\sphericalangle EFD$.

8.3. Assume that the equation has a solution (x_0, y_0, z_0). Then $x_0^2 + 2y_0^2$ is divisible by 7. Since the remainders modulo 7 of the perfect squares are 0, 1, 2 and 4, it follows that both x_0 and y_0 are divisible by 7.

Then the left hand side of the given equation is divisible by 7^2 and hence the number $\underbrace{11\ldots1}_{2005}$ is divisible by 7. But this is a contradiction since 111111 is divisible to 7 and $2005 = 6.334 + 1$.

8.4. a) The eight black circles shown on Fig. 1 have the required property.

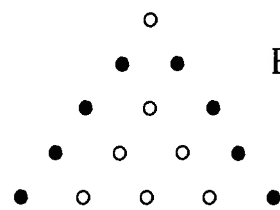


Fig. 1

b) Suppose that it is possible to choose 9 circles such that no three of them are vertices of an equilateral triangle. Then the following three cases are possible.

Case 1. There are no chosen circles which are vertices of the small central triangle. Then we must have at least 4 more non-chosen circles – one in each of the small equilateral triangles at the vertices if the large triangle and one in the triangle with vertices at the midpoints of the sides of the large triangle. Hence we have at least 7 no chosen circles, a contradiction.

Case 2. There is exactly one chosen circle in the central triangle. Without loss of generality we may assume that this is the black circle on Fig. 2. Then, apart from two non-chosen circles in the central triangle, we must have at least two non-chosen circles amongst these denoted by $\star$, at least one amongst these denoted by $\diamond$, at least one amongst these denoted by 1 and at least one amongst the vertices of the large triangle – at least 7 in total, a contradiction.

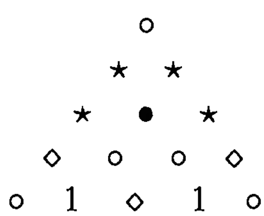


Fig. 2

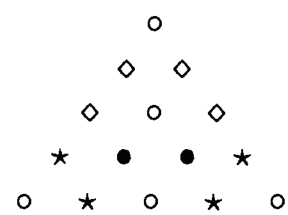


Fig. 3

Case 3. There are exactly two chosen circles in the central triangle. Without loss of generality we may assume that they are the black circles in Fig. 3. Then, apart from the non-chosen circle in the central triangle, we must have at least two non-chosen circles amongst these denoted by $\star$, at least two amongst these denoted by $\diamond$, at least one amongst the vertices of the large triangle and the circle below the two chosen central circles – at least 7 in total, a contradiction.

9.1. a) The discriminant of $f(x)$ is equal to $4a^2 + 13 > 0$.

b) We consecutively have

$$\begin{aligned} -72 &= x_1^3 + x_2^3 = (x_1 + x_2)[(x_1 + x_2)^2 - 3x_1x_2] \\ &= (1 - 2a)(4a^2 - a + 10) = -8a^3 + 6a^2 - 21a + 10. \end{aligned}$$

Therefore the required values of a are the real solutions of the equation

$$8a^3 - 6a^2 + 21a - 82 = 0.$$

Since $a = 2$ is a solution of this equation, we obtain

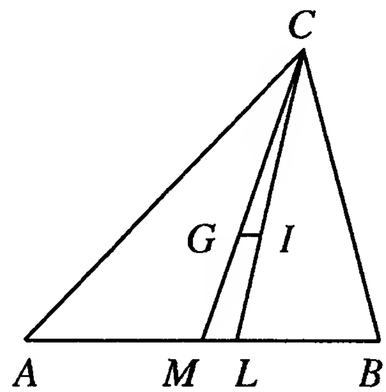
$$8a^3 - 6a^2 + 21a - 82 = (a-2)(8a^2 + 10a + 41).$$

The equation $8a^2 + 10a + 41 = 0$ has no real roots. So the only solution of the problem is $a = 2$.

9.2. Let CM be the median and CL be the bisector of $\triangle ABC$ ($I \in CL$). Using the standard notation for $\triangle ABC$ we have $\frac{AL}{BL} = \frac{AC}{BC} = \frac{b}{a}$, whence $AL = \frac{bc}{a+b}$.

Since AI is the bisector of $\triangle ALC$ through A we get $\frac{CI}{IL} = \frac{CA}{AL} = \frac{a+b}{c}$.

By the Thales theorem we have $\frac{CI}{IL} = \frac{CG}{GM}$, i.e. $a + b = 2c$, whence $a + b = 84$. For $a \le b$ we have $LM = \frac{3}{2}IG = 3$, $AM = 21$, $AL = AM + LM = 24$ and $LB = BM - LM = 18$. Therefore $\frac{b}{a} = \frac{24}{18} = \frac{4}{3}$. Hence $3x + 4x = 84$, i.e. $x = 12$, $AC = 48$ and $BC = 36$. For $b \ge a$ we get $BC = 48$ and $AC = 36$.



9.3. Denote by $S_k^{(m)}$ the money of the k-th player, $k = 1, 2, 3, 4$, after the move and payment of the m-th, $m = 1, 2, 3, 4$, and $S_k^{(0)} = S$ in the beginning, $k = 1, 2, 3, 4$. Denote by a_i the sum of points on the dices thrown by A_i. It follows from the game rules that $S_k^{(m)} = S_k^{(m-1)} + \frac{1}{a_m} S_k^{(m-1)} = S_k^{(m-1)} \frac{1 + a_m}{a_m}$ for $k \ne m$ (i.e. when A_k gets money) and

$$\begin{aligned} S_k^{(k)} &= S_k^{(k-1)} - \frac{1}{a_k} \sum_{i \ne k} S_i^{(k-1)} = S_k^{(k-1)} - \frac{1}{a_k}\left(4S - S_k^{(k-1)}\right) \\ &= S_k^{(k-1)} \frac{1 + a_k}{a_k} - \frac{4S}{a_k}, \end{aligned}$$

when A_k pays.

Using these formulas we find the money of the four players in the end of

the game. We obtain that

$$\begin{aligned}
\frac{6S}{5} &= S_1^{(4)} = PS - \frac{4S(1+a_2)(1+a_3)(1+a_4)}{a_1a_2a_3a_4},\\
\frac{6S}{5} &= S_2^{(4)} = PS - \frac{4S(1+a_3)(1+a_4)}{a_2a_3a_4},\\
\frac{4S}{5} &= S_3^{(4)} = PS - \frac{4S(1+a_4)}{a_3a_4},\\
\frac{4S}{5} &= S_4^{(4)} = PS - \frac{4S}{a_4},
\end{aligned}$$

where

$$P = \frac{(1+a_1)(1+a_2)(1+a_3)(1+a_4)}{a_1a_2a_3a_4}.$$

By the first two equations we get $a_2 = a_1 - 1$, and by the last two we have $a_4 = a_3 - 1$. Now by the second and third equations we obtain

$$\frac{2}{5} = \frac{4}{a_4} - \frac{4(1+a_3)}{a_2a_4} \iff (a_2+10)(10-a_4) = 120.$$

It follows from the later equation that $a_4 < 10$. We also have $a_4 \geq 7$ since the dices are 7 and therefore the minimum sum is 7. It remains to check the possibilities $a_4 = 7$, 8 and 9. The only solution appears for $a_4 = 7$ (the maximum sum is 42) which gives $a_3 = 8$, $a_2 = 30$ and $a_1 = 31$.

9.4. We shall prove that $n+1$, $n+2$ and $n+3$ are prime powers. Assume the contrary and let some of them has the form ab, where $a \geq 2$, $b \geq 2$ and $(a,b) = 1$. Since ab does not divide M, then a or b does not divide M. Let a does not divide M. Then it follows that $a \geq n+1$, i.e. $ab - a \leq 2$. Since $ab - a = a(b-1) \geq 2$ for $a \geq 2$, $b \geq 2$, it follows that $a = 2, b = 2$, which contradicts to $(a,b) = 1$.

Therefore $n+1$, $n+2$ and $n+3$ are prime powers. At least one of them is even, so it has the form 2^x. Analogously, at least one of them is divisible by 3, so it has the form 3^y. By parity arguments we conclude that

$$2^x = 3^y \pm 1.$$

Case 1. Let $2^x = 3^y + 1$. Since $2^x \equiv 1 \pmod 3$ for x even and $2^x \equiv 2 \pmod 3$ for x odd, we see that $x = 2z$, where z is a nonnegative integer, and $(2^z - 1)(2^z + 1) = 3^y$. Then $2^z - 1$ and $2^z + 1$ are powers of 3, which is possible only for $z = 1$. Therefore $3^y = 3$, $2^x = 4$, whence $n = 1$ or $n = 2$. These solutions are achieved for $M = 1$ and $M = 2$, respectively.

Case 2. Let $2^x = 3^y - 1$. We may assume that $x \geq 2$ because for $x = 1$ we obtain one of the above answers for n. We have $3^y \equiv 1 \pmod 4$ for y even and $3^y \equiv 3 \pmod 4$ for y odd. Therefore $y = 2z$ and $2^x = (3^z - 1)(3^z + 1)$. Then $3^z - 1$ and $3^z + 1$ are powers of 2 which is possible only for $z = 1$. Therefore

$3^y = 9$, $2^x = 8$, whence $n = 6$ or $n = 7$. The solution $n = 6$ is achieved, for instance, if $M = 60$. For $n = 7$ we obtain $n + 3 = 10$, which is not a prime power.

Finally, the solutions are $n = 1$, $n = 2$ and $n = 6$.

10.1. For $x \geq 1$ the equation becomes $(x + 6)(5^x - 5^{2-x}) = 0$, whence $x = 2 - x$, т.е. $x = 1$. For $0 \leq x < 1$ the equation becomes an identity, i.e. every $x \in [0, 1)$ is a solution. For $x < 0$ we obtain $2(5^x - 1)(x + 1) = 0$, whence $x = -1$.

The solutions of the problem are $x \in [0, 1] \cup \{-1\}$.

10.2. It is enough to find all reals a such that the converse inequality

$$\sqrt{4 + 3x} < x + a$$

is satisfied for every integer $x \geq -1$. In particular, since $x = 0$ is a solution of $\sqrt{4 + 3x} < x + a$, then $a > 2$ is a necessary condition.

We shall prove that it is also a sufficient condition. Indeed, for $a > 2$ we have $x + a > x + 2$ and it is enough to prove that $x + 2 \geq \sqrt{4 + 3x}$ for every integer $x \geq -1$. We consecutively have

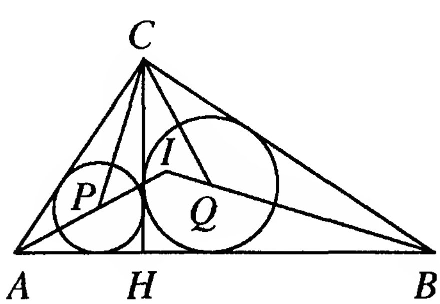
$$x + 2 \geq \sqrt{4 + 3x} \quad \Leftrightarrow \left| \begin{array}{l} (x + 2)^2 \geq 4 + 3x \\ x \geq -\frac{4}{3} \end{array} \right. \quad \Leftrightarrow \left| \begin{array}{l} x^2 + x \geq 0 \\ x \geq -\frac{4}{3} \end{array} \right. .$$

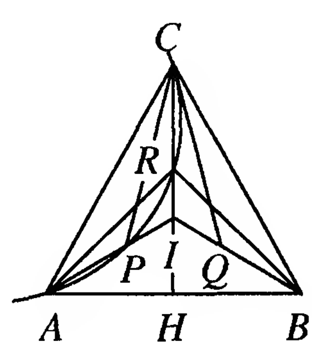
The later system of inequalities is satisfied for $x \in [-\dfrac{4}{3}, -1] \cup [0, \infty)$, i.e. for every integer $x \geq -1$. Therefore the solutions of the problem are $a \in (2, \infty)$.

10.3. ($\Leftarrow$) If $AC = BC$, then the quadrilateral $ABQP$ is cyclic since it is an isosceles trapezoid. If $\sphericalangle ACB = 90^\circ$, then we have $\sphericalangle ACI = \sphericalangle BCI = 45^\circ$, where I is the incenter of $\triangle ABC$. We have also $\sphericalangle APC = \sphericalangle BQC = 135^\circ$, i.e. $\sphericalangle IPC = \sphericalangle IQC = 45^\circ$. Therefore $\triangle IPC \sim \triangle ICA$ and $\triangle IQC \sim \triangle ICB$. Hence

$$IP.IA = IC^2 = IQ.IB$$

and therefore the quadrilateral $ABQP$ is cyclic.





($\Rightarrow$) We consider the circumcircle of $\triangle APC$. If it is tangent to CI at the point C, then $\sphericalangle ACI = \sphericalangle IPC = 45^\circ$ and thus $\sphericalangle ACB = 90^\circ$. Now let us assume that this circle intersects CI again at some point R. Then we have

$IP.IA = IR.IC$ and $IP.IA = IQ.IB$. Therefore $IR.IC = IQ.IB$ and the quadrilateral $BCRQ$ is cyclic. Thus

$$\sphericalangle BRC = \sphericalangle BQC = 135^\circ = \sphericalangle APC = \sphericalangle ARC.$$

Hence $\triangle ARC \cong \triangle BRC$ and $AC = BC$.

10.4. Let q be an integer such that

$$2q^2 \le n < 2(q+1)^2.$$

Then

$$n - 2q^2 < 4q + 2 \le 4\sqrt{\frac{n}{2}} + 2 = 2(\sqrt{2n} + 1).$$

Further, let t be an integer such that $t^2 \le n - 2q^2 < (t+1)^2$. We choose p to be either the number t or $t+1$ depending on the location of $n - 2q^2$ with respect to the midpoint of the interval $[t^2, (t+1)^2]$. More precisely, we set

$$p = \begin{cases} t, & \text{if } n - 2q^2 - t^2 \le t; \\ t+1, & \text{if } n - 2q^2 - t^2 > t. \end{cases}$$

Then we have

$$|p^2 + 2q^2 - n| \le t \le \sqrt{n - 2q^2} \le \sqrt{2(\sqrt{2n} + 1)}.$$

It remains to note that $\sqrt{2(\sqrt{2n} + 1)} \le \sqrt[4]{9n}$ for every $n \ge 160$ and that for $n < 160$ the existence of p and q can be checked directly.

11.1. a) Using the recurrence relation we easily get

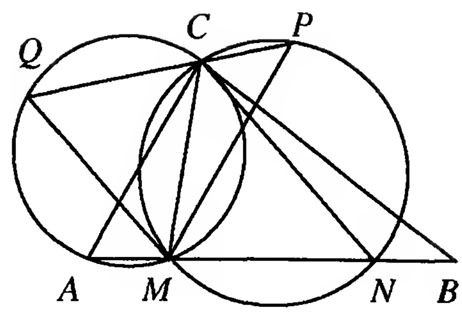
$$\begin{aligned} a_k &= a_{k-1} + 4(k-1) + 3 = a_{k-2} + 4(k-2) + 4(k-1) + 2.3 = \cdots \\ &= a_1 + 4(1 + 2 + \cdots + k - 1) + (k-1).3 = 2k(k-1) + 3(k-1) \\ &= (2k+3)(k-1). \end{aligned}$$

b) We have $\lim\limits_{n\to\infty} \dfrac{\sqrt{a_{kn}}}{n} = \lim\limits_{n\to\infty} \sqrt{(2k + \frac{3}{n})(k - \frac{1}{n})} = \sqrt{2}k$. Therefore the required limit is equal to

$$\frac{1 + 4 + 4^2 + \cdots + 4^{10}}{1 + 2 + 2^2 + \cdots + 2^{10}} = \frac{4^{11} - 1}{3(2^{11} - 1)} = \frac{2^{11} + 1}{3} = 683.$$

11.2. The inequality is defined for $x^2 - x - 2 > 0$ and $3 + 2x - x^2 > 0$, whence $x \in (2, 3)$. Since $x = a + 1$ is a solution, we have $a \in (1, 2)$. Then the inequality is equivalent to $x^2 - x - 2 > 3 + 2x - x^2 \iff (x+1)(2x-5) > 0$ and therefore $x \in \left(\dfrac{5}{2}, 3\right)$.

11.3. a) We have $\sphericalangle QCA = \sphericalangle QMA = \sphericalangle CNA$ and $\sphericalangle PCN = \sphericalangle PMN = \sphericalangle NAC$. Hence



$$\begin{aligned}\sphericalangle QCP &= \sphericalangle QCA + \sphericalangle ACN + \sphericalangle NCP\\ &= \sphericalangle CNA + \sphericalangle ACN + \sphericalangle NAC\\ &= 180^\circ,\end{aligned}$$

i.e. the points P, C and Q are colinear.

b) Set $\sphericalangle ACM = \varphi$, $\sphericalangle NCM = \psi$ and denote by R_1 and R_2 the radii of the circumcircles of ΔAMC and ΔMNC, respectively. Then by the Sine theorem we have

$$\begin{aligned}QC &= 2R_1 \sin \sphericalangle QMC = 2R_1 \sin\psi,\\ CP &= 2R_2 \sin \sphericalangle PMC = 2R_2 \sin\varphi,\\ AM &= 2R_1 \sin\varphi,\\ BM &= 2R_2 \sin\psi.\end{aligned}$$

Therefore $PC.CQ = 4R_1R_2 \sin\varphi \sin\psi = AM.BM$.

As in a) we see that the points L, C and K are colinear. Analogously we have $CK.CL = AN.BN$. Since the circumcircle of ΔMNC passes through K, C and P, the lines PQ and KL do not coincide. Therefore the points P, Q, K and L are cyclic if and only if

$$CP.CQ = CL.CK \iff AM.BM = AN.BN \iff AM = BN.$$

11.4. Let M be an arbitrary positive integer. We shall prove that there exists a term of the sequence $\{a_n\}_{n=1}^{\infty}$, whose decimal representation is obtained from that of M by adding several digits from the right, i.e. the number M is a "beginning" of that member.

Let k be an index such that $a_k \le M.10^l < a_{k+1} < a_k + c$, where l is a positive integer which is greater than the number of the digits of c. Then

$$M.10^l < a_{k+1} < M.10^l + c$$

and obviously a_{k+1} satisfies the above requirement.

Let $m = 2^\alpha 5^\beta t$, where $(t, 10) = 1$. It is enough to prove the assertion of the problem for $m = 10^\gamma t$, where $\gamma = \max\{\alpha, \beta\}$.

Let us consider the number

$$M = 1\underbrace{00\ldots0}_{p}1\underbrace{00\ldots0}_{p}1\ldots1\underbrace{00\ldots0}_{p}1\underbrace{00\ldots0}_{q}.$$

Here $p = k\varphi(t)$, where $\varphi(t)$ is the Euler function, k is a positive integer such that $p > \gamma$, the number q is greater than γ, and the number of 1's is $t+1$.

Then M is a "beginning" of some a_k. Hence the sequence of the digits (formed by the terms of the sequence written one after another) looks like this:

$$f_1 f_2 \dots f_r 1 \underbrace{00\dots0}_{p} 1 \underbrace{00\dots0}_{p} 1 \dots 1 \underbrace{00\dots0}_{p} 1 \underbrace{00\dots0}_{q} \dots,$$

where $f_1, f_2, \dots, f_r$ are the digits before a_k. It is clear now that, depending on the remainder of $\overline{f_1 f_2 \dots f_r 1}$ modulo t, we can add suitable digits from M to $\overline{f_1 f_2 \dots f_r 1}$ in such a way that the resulting number is divisible by $10^\alpha t$.

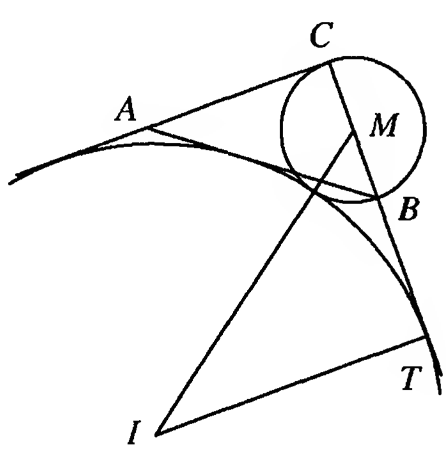
12.1. a) It follows from the Pythagorean theorem that the altitude of $\triangle ABC$ through C is equal to $\sqrt{1-x^2}$. Then

$$r = \frac{S}{p} = \frac{x\sqrt{1-x^2}}{1+x} = x\sqrt{\frac{1-x}{1+x}}.$$

b) We have to find the maximum of the function

$$f(x) = \frac{x^2(1-x)}{1+x}$$

in the interval $(0,1)$. Since $f'(x) = \dfrac{2x(1-x-x^2)}{(x+1)^2}$ the function $f(x)$ increases in the interval $\left(0, \dfrac{\sqrt{5}-1}{2}\right]$ and decreases in the interval $\left[\dfrac{\sqrt{5}-1}{2}, 1\right)$. Therefore the maximum of $f(x)$ in $(0,1)$ is attained for $x = \dfrac{\sqrt{5}-1}{2}$ and is equal to $f\left(\dfrac{\sqrt{5}-1}{2}\right) = \dfrac{5\sqrt{5}-11}{2}$. Hence the maximum possible value of r is $\sqrt{\dfrac{5\sqrt{5}-11}{2}}$.



12.2. Let M be the midpoint of BC, let I be the center of the excircle of $\triangle ABC$ tangent to AB and let T be its tangent point to the line BC. In the standard notation for $\triangle ABC$ we have $IM = \dfrac{a}{2} + r_c$, $IT = r_c$ and $MT = p - \dfrac{a}{2}$ since $BT = p - a$. It follows from the right $\triangle MIT$ that

$$\left(\frac{a}{2} + r_c\right)^2 = r_c^2 + \left(p - \frac{a}{2}\right)^2.$$

Then $ar_c = p(p-a)$. Since $r_c = \dfrac{S}{p-c}$, we obtain by using Heron's formula that

$$aS = p(p-a)(p-c) = \frac{S^2}{p-b},$$

i.e.

$$a(p-b)=S. \tag{1}$$

Since a, b and c form (in this order) an arithmetic progression, we have $a = b-x$, $c=b+x$ and

$$p=\frac{3b}{2},\ p-a=\frac{b}{2}+x,\ p-b=\frac{b}{2},\ p-c=\frac{b}{2}-x.$$

Now by (1) and the Heron formula we obtain the equation

$$(b-x)^2=3\left(\frac{b^2}{4}-x^2\right)$$

which has a unique solution $x=\dfrac{b}{4}$. Therefore $a=\dfrac{3b}{4}$, $c=\dfrac{5b}{4}$ and then $a^2+b^2=c^2$, i.e. $\measuredangle ACB=90°$.

12.3. Subtracting the equalities $a_n+a_{n+1}=2a_{n+2}a_{n+3}+1$ and $a_{n+1}+a_{n+2}=2a_{n+3}a_{n+4}+1$, we get $a_{n+2}-a_n=2a_{n+3}(a_{n+4}-a_{n+2})$. Then it follows by induction on k that

$$a_{n+2}-a_n=2^k a_{n+3}\dots a_{n+2k+1}(a_{n+2k+2}-a_{n+2k}).$$

Hence 2^k divides $a_{n+2}-a_n$ for every k, i.e. $a_{n+2}=a_n$. Therefore $a_{2n-1}=a_1$ and $a_{2n}=a_2$ for every n. Now it follows from the condition of the problem that $(2a_1-1)(2a_2-1)=-4009$. Since $4009=19.211$ and 19 and 211 are primes, we get $2a_1-1=\pm1,\pm19,\pm211,\pm4009$. Therefore there are 8 sequences with the required property.

12.4. We first show that the assertion follows from the following lemma.

LEMMA. *The polynomial $P(z)=z^{2n}+az^{2n-1}+az^{2n-2}+\dots+az+1$ has at least $2n-2$ complex zeros lying on the unit circle and different from ±1.*

Since the coefficients of $P(z)$ are real numbers, its non-real zeros are complex conjugate. It follows from the lemma that there are at least $n-1$ pairs of such zeros and we denote them by $\alpha_1,\overline{\alpha_1},\dots,\alpha_{n-1},\overline{\alpha_{n-1}}$.

We may assume that

$$\begin{aligned} x^2+b_1x+c_1 &= (x-\alpha_1)(x-\overline{\alpha_1}) \\ \dots\quad &\quad \dots \\ x^2+b_{n-1}x+c_{n-1} &= (x-\alpha_{n-1})(x-\overline{\alpha_{n-1}}). \end{aligned}$$

Since $|\alpha_1|^2=\dots=|\alpha_{n-1}|^2=1$ we get $c_1=\dots=c_{n-1}=1$ and therefore $c_n=1$.

Proof of the Lemma. Since the non-real zeros of $P(z)$ are complex conjugate it is enough to prove that this polynomial has at least $n-1$ zeros on the upper unit semicircle.

To do this set $z = e^{i2\theta}$. Then

$$\begin{aligned}\frac{z^{2n}+1}{z^{2n-1}+\cdots+z} &= \frac{(z^{2n}+1)(z-1)}{z(z^{2n-1}-1)} \\ &= \frac{(e^{i2n\theta}+e^{-i2n\theta})(e^{i\theta}-e^{-i\theta})}{(e^{i(2n-1)\theta}-e^{-i(2n-1)\theta})} \\ &= 2\frac{\cos 2n\theta \sin\theta}{\sin(2n-1)\theta} = \frac{\sin(2n+1)\theta}{\sin(2n-1)\theta} - 1.\end{aligned}$$

Therefore we have to prove that the equation

$$f(\theta) = \sin(2n+1)\theta + (a-1)\sin(2n-1)\theta = 0$$

has at least $n-1$ roots in the interval $(0, \frac{\pi}{2})$. This is obvious for $a = 1$ because $f(\theta_k) = 0$, where $\theta_k = \frac{k\pi}{2n+1}$, $1 \le k \le n$.

Observe now that $(k-1)\pi < (2n-1)\theta_k < k\pi$ and this implies that $(-1)^{k-1}\sin(2n-1)\theta_k > 0$. Hence $f(\theta_k)f(\theta_{k+1}) < 0$ for $a \neq 1$ and the Intermediate value theorem implies that the equation $f(\theta) = 0$ has at least one root in each of the intervals $(\theta_1, \theta_2), \ldots, (\theta_{n-1}, \theta_n)$. This completes the proof of the lemma.

Remark. If follows from the above that the polynomial $P(z)$ has at most two real zeros x_1 and x_2 (possibly $x_1 = x_2$) and $x_1x_2 = 1$. It can be proved that:

1) if $a > 2$, then $x_1 < -1$ and $x_2 \in (-1, 0)$;
2) if $a = 2$, then $x_1 = x_2 = -1$;
3) if $a < -\frac{2}{2n-1}$, then $x_1 > 1$ and $x_2 \in (0, 1)$;
4) if $a = -\frac{2}{2n-1}$, then $x_1 = x_2 = 1$;
5) if $a \in \left(-\frac{2}{2n-1}, 2\right)$, then the polynomial $P(z)$ has no real zeros. In this case it has a zero $z = e^{i2\theta}$, where $\theta \in (0, \theta_1)$ for $a \in \left(-\frac{2}{2n-1}, 1\right)$ and $\theta \in \left(\theta_n, \frac{\pi}{2}\right)$ for $a \in (1, 2)$.

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9.1. It follows from the condition that there is a polynomial $q(x)$ of degree 2 such that

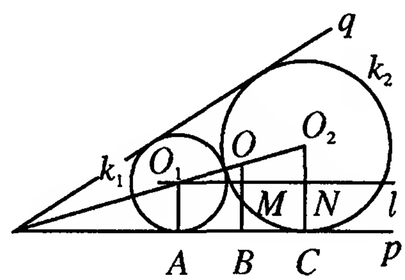
$$x^4 - 3ax^3 + ax + b = q(x)(x^2-1) + (a^2+1)x + 3b^2$$

for every x. Setting $x = 1$ and $x = -1$ we obtain the system

$$\left| \begin{array}{ll} a^2 + 3b^2 + 2a - b & = 0 \\ a^2 - 3b^2 + 2a + b + 2 & = 0 \end{array} \right.$$

and it is easy to see that $a = -1$, $b = \dfrac{1 \pm \sqrt{13}}{6}$. A direct verification shows that these values of a and b are solutions indeed.

9.2. Denote the circles by $k_1(O_1, r)$, $k_2(O_2, R)$ and $k(O, x)$, where $r < x < R$. Let A, B and C be the feet of the perpendiculars from O_1, O and O_2, respectively, to the arm p of the given angle Opq. Let l be the line through O_1 parallel to p and let l meet OB and O_2C at points M and N, respectively. Then $\triangle O_1OM \sim \triangle O_1O_2N$ and therefore



$$\frac{OO_1}{O_1O_2} = \frac{OM}{O_2N}.$$

We have $OM = OB - BM = OB - O_1A = x - r$, $O_1O_2 = r + R$ and $O_2N = O_2C - CN = O_2C - O_1A = R - r$.

If k passes through O_1, then $OO_1 = x$ and we get the equation

$$\frac{x}{R+r} = \frac{x-r}{R-r},$$

whence $x = \dfrac{r+R}{2}$. If k passes through O_2, then $OO_1 = R + r - x$ and

$$\frac{R+r-x}{R+r} = \frac{x-r}{R-r},$$

whence we have again $x = \dfrac{r+R}{2}$.

In both cases O is the midpoint of O_1O_2 and k passes through O_1 and O_2.

9.3. Assume that the pair $(x, x+1)$ is a solution of the equation. Then we have

$$\begin{aligned} a - b &= a^k x - b^k(x+1) \Leftrightarrow \\ b^k &= (a-b)\Big[x(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1}) - 1\Big]. \end{aligned}$$

Suppose that the numbers $a-b$ and $x(a^{k-1}+a^{k-2}b+\cdots+ab^{k-2}+b^{k-1})-1$ have a common prime divisor p. Then p divides b^k, i.e. p divides b. But p divides also $a-b$ and therefore it divides a as well. Now the fact that p divides $x(a^{k-1}+a^{k-2}b+\cdots+ab^{k-2}+b^{k-1})-1$ implies that p divides 1, a contradiction.

Therefore the numbers $a-b$ and $x(a^{k-1}+a^{k-2}b+\cdots+ab^{k-2}+b^{k-1})-1$ are co-prime, which implies that each of them is a k-th power (up to a sign). In particular, $|a-b|$ is a k-th power.

The case of a solution $(x+1,x)$ is considered analogously.

9.4. Answer: $p=2$. The condition $p>1$ is necessary (but not sufficient!) for existence of four roots. We consider two cases:

Case 1. If $x^2-px-2p+1=p-1 \iff x^2-px-3p+2=0$ then by the Vieta theorem we obtain $x_1^2+x_2^2=p^2-2(2-3p)=p^2+6p-4$.

Case 2. If $x^2-px-2p+1=1-p \iff x^2-px-p=0$ then by the Vieta theorem we obtain $x_3^2+x_4^2=p^2+2p$.

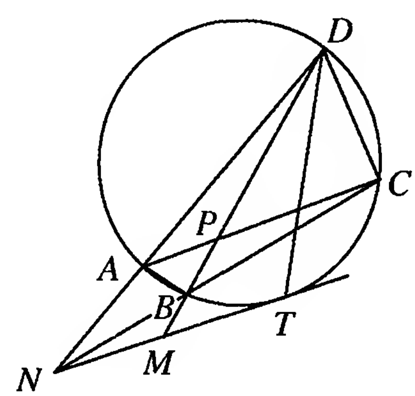
Now the condition implies

$$x_1^2+x_2^2+x_3^2+x_4^2=20 \iff 2p^2+8p-4=20 \iff p^2+4p-12=0,$$

whence $p=2$ or $p=-6$. The second value does not satisfy $p>1$. For $p=2$ we do have four real roots (direct check!).

9.5. It follows from the condition that T lies on the arc $\overset{\frown}{BC}$. Let $M=NT\cap DP$ be the midpoint of NT. Then we have

$$MB.MD=MT^2=MN^2.$$



Thus $MB:MN=MN:MD$ and it follows that $\triangle NMB \sim \triangle DMN$. Hence

$$\sphericalangle MNB=\sphericalangle MDN=\sphericalangle NCA,$$

i.e. $NT||AC$. Thus

$$\frac{NT}{AP}=\frac{2NM}{AP}=\frac{2MD}{PD}=3.$$

9.6. The number of all pairs of players is $\dfrac{25.24}{2}=300$ and after each game 10 of them become impossible. Therefore at most $300:10=30$ games are possible.

We shall prove that 30 games are possible. We denote the pairs of players by (m,n), where $1\le m,n\le 5$ are integers (in other words, we put them in a table 5×5).

In the game i, $1\le i\le 5$, we put the five pairs with $m=i$ (i.e. those from the i-th row of the table). In the game $6+5k+i$, $0\le i\le 4$, $0\le k\le 4$, we set

the pair (m,n) such that $mk+n$ is congruent to i modulo 5. It is clear that for any fixed values of k,i,m there exists a unique n such that $mk+n \equiv i \pmod 5$. Thus we have one pair in each row, i.e. the pairs are five and they have not played in the first 5 games.

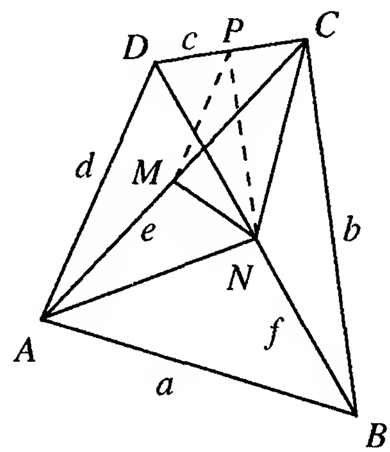
For every two pairs (m,n) and (m',n'), $m' \neq m$, the numbers $k(m-m')$, $k=0,1,2,3,4$, give different remainders modulo 5. Hence there exists a unique k such that $k(m-m') \equiv n-n' \pmod 5$. Equivalently, $km-n$ and $km'-n'$ have the same remainder i modulo 5 and the numbers k and i determine the unique game in which the pairs (m,n) and (m',n') participate.

10.1. Set $u = 2^x > 0$ and $v = 3^y > 0$. Then the system becomes

$$\left| \begin{array}{rcrcrcr} 3u^2 & + & 2uv & - & v^2 & = & 0 \\ 2u^2 & - & 5uv & + & v^2 & = & -8 \end{array} \right. .$$

The first equation can be written as $(u+v)(3u-v)=0$, whence $u=-v$ or $3u=v$. The first case is not possible since u and v are positive. Plugging $v=3u$ in the second equation gives $u^2=2$, whence $u=\sqrt{2}$ and $v=3\sqrt{2}$. Thus $x=\dfrac{1}{2}$ and $y=1+\dfrac{1}{2}\log_3 2$.

10.2. a) Denote the midpoints of AC and BD by M and N, respectively. Using the median formula for $\triangle BMD$ we have



$$MN^2 = \frac{2MB^2 + 2MD^2 - f^2}{4}.$$

We also have

$$MB^2 = \frac{2a^2+2b^2-e^2}{4}, \quad MD^2 = \frac{2c^2+2d^2-e^2}{4}$$

and plugging these expressions in the equality for MN^2 gives

$$0 \le 4MN^2 = a^2+b^2+c^2+d^2-e^2-f^2.$$

b) The Ptolemy's theorem states that $ac+bd=ef$. Using this we write the identity from a) as

$$(a-c)^2+(b-d)^2 = (e-f)^2+4MN^2.$$

In order to prove the required inequality it is enough to show that the inequality $(b-d)^2 \le 4MN^2$ holds. But it follows from the triangle inequality for $\triangle MNP$, where P is the midpoint of CD.

10.3. The left hand side of the given equation is a multiple of m, n and $m-n$. Therefore $m=2^a$, $n=2^b$ and $m-n=2^c$ for some nonnegative integers a, b and c, where $a>b$. It is obvious that $2^b(2^{a-b}-1)=2^c$, whence $a-b=1$.

Plugging $b = a - 1$ in the given equation we obtain

$$\begin{aligned}[2^{2a} + 2^{2a-1}, 2^{2a-1} - 2^{2a-2}] + [2^a - 2^{a-1}, 2^{2a-1}] &= 2^{2a-1} + 3 \cdot 2^{2a-1} \\ &= 2^{2a+1} = 2^{2005}.\end{aligned}$$

Hence $a = 1002$, $m = 2^{1002}$ and $n = 2^{1001}$.

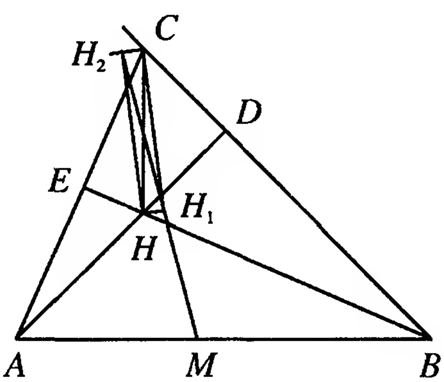
10.4. The first equation is quadratic with discriminant $D = (9a^2 - 1)^2$. Therefore it has two different solutions for $a \neq \pm\frac{1}{3}$ and exactly one solution for $a = \pm\frac{1}{3}$.

Since the function

$$2x^3 + 6x + (3a - 1)^2 12^x$$

is strictly increasing, the second equation has at most one solution. For $a = \frac{1}{3}$ it has solution $x = 0$, while for $a = -\frac{1}{3}$ it is not defined. Finally, the only value of a which satisfies the condition, is $a = \frac{1}{3}$.

10.5. Denote by D and E the feet of the altitudes of $\triangle ABC$ from the vertices A and B, respectively. The quadrilateral $HDCE$ is inscribed in the circle with diameter CH. The points H_1 and H_2 are the midpoints of the two arcs $\overset{\frown}{DE}$ since CH_1 and CH_2 are the internal and external bisectors of $\measuredangle ACB$, respectively. Hence the line H_1H_2 is the perpendicular bisector of the segment DE.



On the other hand, the quadrilateral $ABDE$ is also inscribed, this time in the circle with diameter AB. Therefore the perpendicular bisector of the chord DE passes through the center M of the circumcircle of $ABDE$.

10.6. Let $x_1, x_2, \ldots, x_{1000}$ be an arbitrary rearrangement of the numbers $1, 2, \ldots, 1000$. Set

$$S_1 = x_1 + x_2 + \cdots + x_{50}, \ldots, S_{20} = x_{951} + x_{952} + \cdots + x_{1000}.$$

Since $S_1 + \cdots + S_{20} = 500500$, we have $S_i \geq \frac{500500}{20} = 25025$ for at least one index i.

On the other hand, if a number B has the required property then we have $B \leq 25025$. To see this consider the rearrangement

$$1000, 1, 999, 2, \ldots, 501, 500$$

and take arbitrarily fifty consecutive numbers in it. If the first number is greater than 500, then the sum of these fifty numbers is 25025, otherwise it is 25000. Hence $A = 25025$.

11.1. We write the given equation as

$$(1) \qquad 2ay^2 - \sqrt{2}(a-3)y + 1 = 0,$$

where

$$y = \frac{\sqrt{2}}{2}(\sin x + \cos x) = \sin(x + 45^\circ) \in [-1, 1].$$

It has a solution if and only if (1) has a solution in the interval $[-1, 1]$.

For $a = 0$ the equation (1) is linear and its root $y = -\dfrac{\sqrt{2}}{6}$ belongs to the interval $[-1, 1]$.

Let $a \neq 0$ and set $f(y) = 2ay^2 - \sqrt{2}(a-3)y + 1$. We have $f(1) = 0$ and $f(-1) = 0$ for $a = -\dfrac{7\sqrt{2}+8}{2}$ and $a = \dfrac{7\sqrt{2}-8}{2}$, respectively. The quadratic polynomial $f(y)$ has exactly one root in $(-1, 1)$ if and only if $f(-1)f(1) < 0$. This implies that $a \in \left(-\dfrac{7\sqrt{2}+8}{2}, 0\right) \cup \left(0, \dfrac{7\sqrt{2}-8}{2}\right)$.

Further, $f(y)$ has two roots in $(-1, 1)$ if and only if

$$\left|\begin{array}{l} af(-1) > 0 \\ af(1) > 0 \\ D = 2a^2 - 20a + 18 \geq 0 \\ -1 < \dfrac{\sqrt{2}(a-3)}{4a} < 1 \end{array}\right. \iff \left|\begin{array}{l} a \in (-\infty, 0) \cup \left(\dfrac{7\sqrt{2}-8}{2}, \infty\right) \\ a \in \left(-\infty, -\dfrac{7\sqrt{2}+8}{2}\right) \cup (0, \infty) \\ a \in (-\infty, 1] \cup [9, \infty) \\ a \in \left(-\infty, -\dfrac{3+6\sqrt{2}}{7}\right) \cup \left(\dfrac{6\sqrt{2}-3}{7}, \infty\right) \end{array}\right. .$$

Hence $a \in \left(-\infty, -\dfrac{7\sqrt{2}+8}{2}\right) \cup \left(\dfrac{7\sqrt{2}-8}{2}, 1\right] \cup [9, \infty)$.

Taking into account the above cases we get $a \in (-\infty, 1] \cup [9, \infty)$.

Second Solution. The discriminant of $f(y)$ is non-negative if and only if $a \in (-\infty, 1] \cup [9, \infty)$.

For $a \leq 1$ we have $f(0) = 1 > 0$ and $f\left(-\dfrac{1}{\sqrt{2}}\right) = 2a - 2 \leq 0$, which implies that the equation $f(y) = 0$ has a root in the interval $\left(-\dfrac{1}{\sqrt{2}}, 0\right) \subset (-1, 1)$.

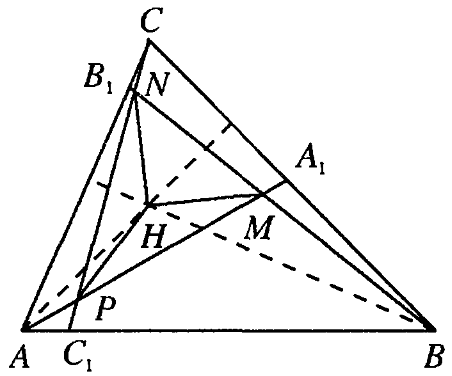
For $a \geq 9$ we have $f(0) = 1 > 0$ and $f\left(\dfrac{1}{3\sqrt{2}}\right) = \dfrac{2}{9}(9 - a) \leq 0$, which shows that the equation $f(y) = 0$ has a root in the interval $\left(0, \dfrac{1}{3\sqrt{2}}\right) \subset (-1, 1)$.

Hence the given equation has a solution if and only if $a \in (-\infty, 1] \cup [9, \infty)$.

11.2. Set $AA_1 \cap BB_1 = M$, $BB_1 \cap CC_1 = N$ and $CC_1 \cap AA_1 = P$. Using the standard notation for the angles of $\triangle ABC$, we have

$$\measuredangle PMN = \varphi - \measuredangle B_1BC = \varphi - (\varphi - \gamma) = \gamma.$$

Analogously, we get $\sphericalangle MNP = \alpha$ and $\sphericalangle NPM = \beta$, i.e. $\Delta NPM \sim \Delta ABC$.



Let H be the orthocenter of ΔABC. The equalities $\sphericalangle HCC_1 = \sphericalangle HBB_1 = \sphericalangle HAA_1 = 90^\circ - \varphi$ imply that each of the quadrilaterals $ABMH$, $BCNH$ and $ACHP$ is cyclic. Therefore $\sphericalangle HMA = \sphericalangle HBA = 90^\circ - \alpha$ and $\sphericalangle HPM = 180^\circ - \sphericalangle APH = \sphericalangle ACH = 90^\circ - \alpha$, which implies that H is the circumcenter of ΔMNP.

b) We have proved that the points A, B, M and H are cyclic. Then the Sine theorem gives

$$\frac{MH}{\sin(90^\circ - \varphi)} = \frac{c}{\sin(180^\circ - \gamma)},$$

i.e. $MH = 2R\cos\varphi$. Since MH is the circumradius of ΔMNP, we conclude that

$$2 - \sqrt{3} = \frac{S_{MNP}}{S_{ABC}} = \frac{MH^2}{R^2} = 4\cos^2\varphi \iff 2\cos 2\varphi = -\sqrt{3}.$$

Noting that $0 < 2\varphi < 180^\circ$ we get $2\varphi = 150^\circ$, i.e. $\varphi = 75^\circ$.

11.3. a) A direct check shows that the condition is satisfied when $a+b+c = d$. Let us assume that $a+b+c > d$. Then it is easy to see that

$$ab + cd > (d-a)(d-b).$$

We have analogously $bc+ad > (d-b)(d-c)$ and $ac+bd > (d-a)(d-c)$. Now the multiplication of these three inequalities gives a contradiction. Analogous arguments lead to a contradiction when $a+b+c < d$ and therefore $d = a+b+c$.

b) For a fixed d, $3 \le d \le n$, the equation $d = a+b+c$ has

$$\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$$

solutions. (This can be proved as follows. Write consecutively d 1's. Then the number of the solutions is equal to the number of the ways one can put two separating lines in that sequence; for example $111|11\dots11|1$ corresponds to $a = 3$, $b = d-4$, $c = 1$.) This formula is true also for $d = 1$ and $d = 2$ since the equation has no solutions in these cases.

It remains to calculate

$$\begin{aligned}\sum_{d=1}^{n}\frac{(d-1)(d-2)}{2} &= \frac{1}{2}\sum_{d=1}^{n} d^2 - \frac{3}{2}\sum_{d=1}^{n} d + n \\ &= \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{3}{2}\cdot\frac{n(n+1)}{2} + n \\ &= \frac{n(n-1)(n-2)}{6}.\end{aligned}$$

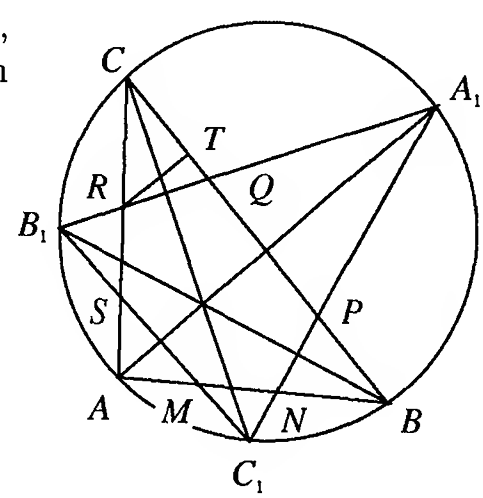
11.4. Since $ax > 0$ and $4^x - 3^x > 0$ it follows that $a > 0$ and $x > 0$. For $a > 0$, $x > 0$ and $ax \neq 1$ the equation is equivalent to

$$3^x + 4^x = 7.2^{x-1}\sqrt{4^x - 3^x} \iff 45\left(\frac{4}{3}\right)^{2x} - 57\left(\frac{4}{3}\right)^x - 4 = 0.$$

Setting $y = \left(\frac{4}{3}\right)^x > 0$ we obtain the equation $45y^2 - 57y - 4 = 0$ with solutions $y_1 = \frac{4}{3}$ and $y_2 = -\frac{1}{15}$. Hence $x = 1$. The condition $ax \neq 1$ now implies that $a \neq 1$

The required values of a are $a \in (0, +\infty) \setminus \{1\}$.

11.5. a) If RT is the altitude of $\triangle CRQ$, then $RT = CR\sin\gamma$. Using the Sine theorem for $\triangle B_1RC$ we get



$$CR = \frac{B_1C\sin\frac{\alpha}{2}}{\cos\frac{\gamma}{2}} = \frac{2R\sin\frac{\beta}{2}\sin\frac{\alpha}{2}}{\cos\frac{\gamma}{2}},$$

where R is the circumradius of $\triangle ABC$. Hence

$$RT = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = r,$$

where r is the inradius of $\triangle ABC$.

b) As in a) we see that the altitude of $\triangle BNP$ through N is equal to r, which means that the lines NR and BC are parallel and the distance between them is r. Therefore $I \in NR$, where I is incenter of $\triangle ABC$.

The same argument shows that $I \in MQ$ and $I \in SR$.

11.6. We first prove the following lemma.

LEMMA. *For any five vertices of a regular 13-gon there exists an isosceles triangle with vertices amongst these points.*

Proof of the lemma. Let the five points form a convex pentagon $ABCDE$. We first consider the case when there exist two pairs of parallel lines determined by some vertices of $ABCDE$. We have the following possibilities:

1) There are two pairs of parallel sides of $ABCDE$ – for example $AE||CD$ and $BC||DE$. Then we have $\sphericalangle AED = \sphericalangle BCD$ or $\sphericalangle AED = 180^\circ - \sphericalangle BCD$, i.e. $\sin\sphericalangle AED = \sin\sphericalangle BCD$. Hence $AD = 2R\sin\sphericalangle AED = 2R\sin\sphericalangle BCD = BD$ and therefore the triangle $\triangle ABD$ is isosceles.

2) There are two diagonals that are parallel respectively to two sides of $ABCDE$. Without loos of generality we may assume that $AB||CE$. If $AC||DE$ we conclude as in 1) that $BC = CD$. If $AD||BC$, then $EB = BD$. The cases $BE||CD$ and $BD||AE$ are similar.

3) There is a diagonal of $ABCDE$ that is parallel to its side and a pair of parallel sides of $ABCDE$. Without loos of generality we may assume that

$AB||CE$. Since AE is not parallel to BC (otherwise $ABCE$ is a rectangle with vertices amongst the vertices of a regular 13-gon) we may assume that $DE||BC$. Then $DC = CA$.

Let us now assume that there exists at most one pair of parallel lines determined by some vertices of $ABCDE$. Since the vertices of $ABCDE$ determine 10 lines, at least 9 of them are not parallel to each other. Consider the 9 pairs of vertices determining such lines. Every such pair is a base of an isosceles triangle whose third vertex is a vertex of the 13-gon. Note that all such third vertices are different because no two of the bases are parallel. Hence at least one of these 9 vertices is a vertex of $ABCDE$ and this proves the existence of the desired isosceles triangle. This completes the proof of the lemma.

The regular 26-gon is formed by two disjoint regular 13-gons. If we choose 9 of its vertices then at least 5 of them are vertices of one of these 13-gons. Then it follows from the lemma that there exists an isosceles triangle with vertices amongst the vertices of this 13-gon.

Finally, note that there exist 8 vertices such that no three of them form an isosceles triangle. For example, if the vertices are labelled from 1 to 26, then we choose the vertices 1, 2, 4, 5, 10, 11, 13 and 14.

12.1. Set

$$\frac{a(a-b)+b(b-c)+c(c-a)}{2} = d^2,$$

where d is an integer, $x = a-b$, $y = b-c$ and $z = c-a$. Then we have

$$x+y+z=0,\ x^2+y^2+z^2=4d^2. \tag{1}$$

Since any square is congruent to 0 or 1 modulo 4, it follows from (1) that the integers x, y and z are even. Set $x_1 = \frac{x}{2}$, $y_1 = \frac{y}{2}$ and $z_1 = \frac{z}{2}$. Then (1) gives

$$x_1+y_1+z_1=0,\ x_1^2+y_1^2+z_1^2=d^2$$

and we conclude as above that x_1, y_1, z_1 and d are even integers. Repeating the same argument we see that 2^n divides x, y and z for every positive integer n. Therefore $x=y=z=0$, i.e. $a=b=c$.

12.2. The first condition of the problem is equivalent to the assertion that the equation $x^3+ax+b=0$ has a double real root $x_1 \neq 0$ and a simple real root $x_2 \neq 0$, where $x_2 \neq x_1$. Therefore

$$x^3+ax+b=(x-x_1)^2(x-x_2).$$

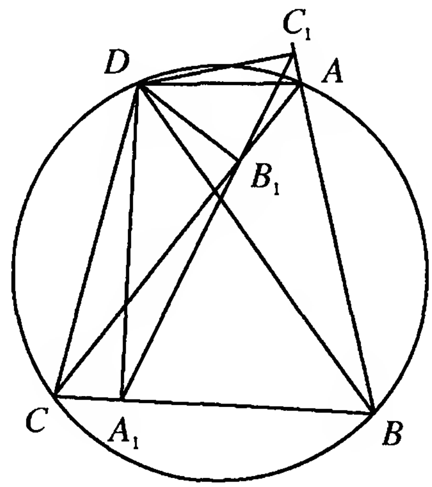
We also have $\sphericalangle ACB = 90°$, where $A=(x_1,0)$, $B=(x_2,0)$, $C=(0,b)$ and $x_1x_2<0$. Hence $AO.BO = CO^2$, i.e. $-x_1x_2=b^2$. Since $b=-x_1^2x_2$ and $x_1, x_2 \neq 0$, we get $x_1^3x_2=-1$. On the other hand, we have $2x_1+x_2=0$ and therefore $2x_1 = -x_2 = \frac{1}{x_1^3}$. Hence $x_1 = \pm\frac{1}{\sqrt[4]{2}}$ and $x_2 = \mp\sqrt[4]{8}$. Then

$a = x_1^2 + 2x_1x_2 = -\dfrac{3}{\sqrt{2}}$ and $b = -x_1^2x_2 = \pm\sqrt[4]{2}$. The above arguments imply as well that these two values of b are solutions indeed.

12.3. Let $ABCD$ be a cyclic quadrilateral. Then the Simson theorem for $\triangle ABC$ gives $DB_1 \perp AC$. Hence $\sphericalangle B_1C_1D = \sphericalangle B_1AD = \sphericalangle CBD$, $\sphericalangle B_1DC_1 = \sphericalangle B_1AC_1 = \sphericalangle CDB$ and therefore $\triangle B_1C_1D \sim \triangle CBD$.

Analogously $\triangle B_1A_1D \sim \triangle ABD$, whence

$$DA : DB : DC = \frac{1}{DA_1} : \frac{1}{DB_1} : \frac{1}{DC_1}.$$



This together with the Ptolemy's theorem for $ABCD$ gives

$$\text{(1)} \qquad \frac{BC}{DA_1} + \frac{BA}{DC_1} = \frac{AC}{DB_1}.$$

Conversely, suppose that the identity (1) is true. Set $x = \dfrac{DB_1}{DA_1}$ and $y = \dfrac{DB_1}{DC_1}$. Squaring (1) and applying the Cosine theorem for $\triangle ABC$, we see that the ratio $\dfrac{BA}{BC}$ is a root of the equation

$$\text{(2)} \qquad (y^2 - 1)t^2 + 2(xy + \cos \sphericalangle ABC)t + x^2 - 1 = 0.$$

Since the point B_1 lies on the segment A_1C_1, the inequality $DB_1 \geq DA_1$ implies that $DB_1 < DC_1$. Hence $x \geq 1$ and $0 < y < 1$, which shows that (2) has at most one positive root.

On the other hand, it is easy to see that C_1 and A_1 lie on the open rays $BA^{\rightarrow}$ and $BC^{\rightarrow}$, and the line through B_1 perpendicular to DB_1 intersects these two rays. Denote these intersection points by A' and C'. Then the converse Simson theorem implies that the convex quadrilateral $A'BC'D$ is cyclic. Hence the identity (1) for $A'BC'D$ is satisfied, i.e. $\dfrac{BA'}{BC'}$ is a root of the equation (2).

Therefore $\dfrac{BA}{BC} = \dfrac{BA'}{BC'}$, i.e. $AC || A'C'$. But the lines AC and $A'C'$ have a common point B_1 and this shows that $A = A'$ and $C = C'$.

Remark. We have used the condition $DB_1 \geq DA_1$ only in the proof that the condition (1) is sufficient.

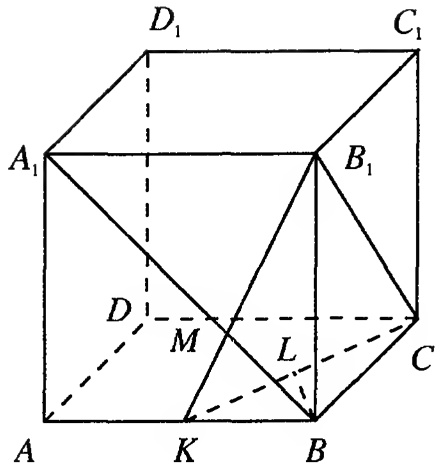
through B is equal to $h = BM \sin 60° = \dfrac{x\sqrt{6}}{2(x+1)}$. Since $B_1K = CK = \sqrt{x^2+1}$ and $B_1C = \sqrt{2}$, we have $S_{B_1KC} = \dfrac{\sqrt{2x^2+1}}{2}$. Therefore

$$V = \frac{h.S_{B_1KC}}{3} = \frac{x\sqrt{6(2x^2+1)}}{12(x+1)}.$$

12.4. We may assume that the edges of the cube have length 1. Denote $M = A_1B \cap KB_1$ and set $KB = x$. Since $\Delta KBM \sim \Delta A_1MB_1$ it follows that $\dfrac{MB}{A_1M} = x$. Now using the identity $A_1M + MB = \sqrt{2}$, we get $MB = \dfrac{x\sqrt{2}}{x+1}$. Denote by V the volume of the tetrahedron $KBCB_1$. Then

$$V = \frac{1}{3}BB_1.S_{KBC} = \frac{x}{6}. \qquad (1)$$

On the other hand, the altitude of $KBCB_1$



This and (1) imply that

$$\frac{x}{6} = \frac{x\sqrt{6(2x^2+1)}}{12(x+1)}$$

and we obtain easily that $x = \dfrac{1}{2}$.

Denote by L the foot of the perpendicular from B to KC. Then $KC \perp BL$ and $KC \perp BB_1$ which shows that $KC \perp B_1L$. Hence $\alpha = \sphericalangle B_1LB$. We get from ΔKBC that $BL = \dfrac{1}{\sqrt{5}}$ and therefore $\tan\alpha = \sqrt{5}$.

12.5. Suppose that it is not possible to cut a triangle T of area $\sqrt{3}$ from the band. Then it is clear that the lengths of its altitudes are greater than $\sqrt{3}$. Hence the lengths of its sides are less than 2.

Let α be the smallest angle of T. Then $\alpha \leq 60^\circ$ and we get

$$\sqrt{3} = \frac{bc\sin\alpha}{2} < \frac{2.2\sin 60^\circ}{2} = \sqrt{3},$$

a contradiction.

12.6. a) Let $n \in A$ and $O_n = \{n, f(n), -n, f(-n)\}$. Since $f(f(n)) = -n$ and $f(f(-n)) = n$, it follows easily that if $k \in A$ then either $O_k = O_n$ or $O_n \cap O_k = \phi$. Moreover, we obtain $f(n) \neq f(-n)$ for $n \neq 0$.

Further, if $f(\pm n) = \pm n$, then $\mp n = f(f(\pm n)) = f(\pm n) = \pm n$, i.e. $n = 0$. Also, $f(\pm n) = \mp n$ gives $\mp n = f(f(\pm n) = f(\mp n)$ and then $n = 0$. Therefore $|O_n| = 4$ for $n \neq 0$ which means that $A \setminus \{0\}$ splits into disjoint quadruples. In particular, the number m is even.

b) Let $m = 2k$ and $f : A \to A$ be a function with the desired property. Set $A_+ = \{1, 2, \ldots, m\}$. We note that $f(-n) = f(f(f(n)) = -f(n)$ and, in particular, $f(0) = 0$. Hence for $n \neq 0$ either $f(n) > 0$ or $f(-n) < 0$. This means that the quadruple O_n is uniquely determined by a pair $(n', f(n'))$ of distinct numbers from A_+ – $(n, f(n))$ or $(f(-n), n)$. Therefore f induces a pairing of A_+ into ordered pairs.

Conversely, any pairing of A_+ into ordered pairs (n, k) defines a function with the required properties by setting

$$f(0) = 0, \; f(n) = k, \; f(k) = -n, \; f(-n) = -k, \; f(-k) = n.$$

It remains to count the number of the pairings of A_+ into ordered pairs. Ordering all pairs of a given pairing one after another (this can be done in $k!$ ways) we obtain a permutation of the numbers $1, 2, \ldots, m$. This gives classes of "equivalent" permutations of $k!$ elements. Therefore the required number is equal to $\frac{m!}{k!}$.

54. Bulgarian Mathematical Olympiad
National Round

1. We first prove the following lemma.

LEMMA. *If p, q, r and $\sqrt{p}+\sqrt{q}+\sqrt{r}$ are rational numbers then $\sqrt{p}$, $\sqrt{q}$ and $\sqrt{r}$ are also rational numbers.*

Proof of the lemma. Let $\sqrt{p}+\sqrt{q}+\sqrt{r}=s$, where $pqr \neq 0$ and s is a rational number. Then $\sqrt{p}+\sqrt{q}=s-\sqrt{r}$ and we get by squaring that

$$p+q+2\sqrt{pq}=s^2+r-2s\sqrt{r} \iff 2\sqrt{pq}=s^2+r-p-q-2s\sqrt{r}.$$

Squaring the last identity gives $4pq = M^2+4s^2r-4Ms\sqrt{r}$, where $M = s^2+r-p-q>0$. Therefore $\sqrt{r}$ is rational and we see in the same way that $\sqrt{p}$ and $\sqrt{q}$ are rational. This completes the proof of the lemma.

Let the positive integers x, y and z have the required property. Then the lemma implies that $\sqrt{\dfrac{2005}{x+y}}$, $\sqrt{\dfrac{2005}{x+z}}$ and $\sqrt{\dfrac{2005}{y+z}}$ are rational numbers. Set $\sqrt{\dfrac{2005}{x+y}}=\dfrac{a}{b}$, where a and b are coprime positive integers. Then $2005b^2 = (x+y)a^2$ and it follows that a^2 divides 2005. Hence $a = 1$ and therefore $x+y=2005b^2$. In the same way we obtain $x+z=2005c^2$ and $y+z=2005d^2$, where c and d are positive integers. Then

$$\sqrt{\frac{2005}{x+y}}+\sqrt{\frac{2005}{x+z}}+\sqrt{\frac{2005}{y+z}}=\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

is a positive integer. Since b, c and d are positive integers, we have

$$1 \le \frac{1}{b}+\frac{1}{c}+\frac{1}{d} \le 3.$$

If $\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=3$, then $b=c=d=1$ and the system $x+y=x+z=y+z=2005$ has no solutions in positive integers.

If $\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=2$, then one of the numbers b, c and d is equal to 1 and the other two are equal to 2. Again, the system for x, y and z has no solutions in positive integers.

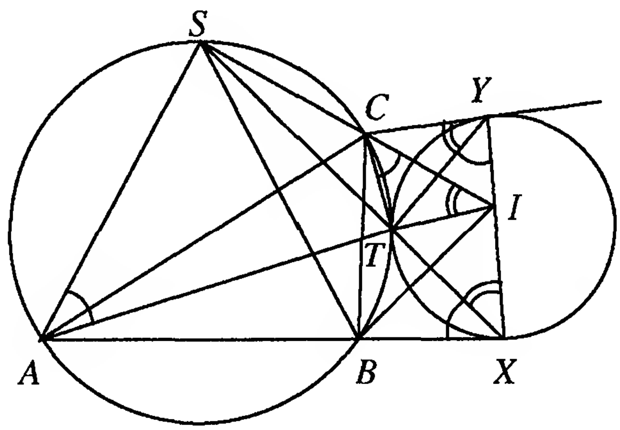
It remains to consider the case $\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=1$. Suppose that $b \ge c \ge d > 1$. Then $\dfrac{3}{d} \ge 1$ and therefore $d=2$ or $d=3$. If $d=3$ we get $b=c=3$, and if $d=2$ we have $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{2}$. This equation has two solutions: $b=3$, $c=6$ and $b=c=4$.

The inspection shows that the system for x, y and z has a solution in positive integers only when $d=2$, $b=c=4$ and in this case $x=14.2005$, $y=z=2.2005$. Therefore the solutions of the problem are all triples (x,y,z) in which two numbers are equal to 2.2005 and the third one is equal to 14.2005.

2. a) Since the circles k_1 and k_2 are tangent at the point T, we have

$$\sphericalangle BXT = \frac{\overset{\frown}{XT}}{2} = \frac{\overset{\frown}{TS}}{2} = \sphericalangle TAS.$$

Then it follows easily that S is the midpoint of the arc $\overset{\frown}{AB}$, i.e. $SA = SB$. Hence $\sphericalangle TCI = \sphericalangle TAS$ (the quadrilateral $ATCS$ is cyclic), $\sphericalangle TAS = \sphericalangle BXT$ and $\sphericalangle BXT = \sphericalangle TYX$. Therefore $\sphericalangle TCI = \sphericalangle TYI$, which shows that the quadrilateral $CTIY$ is cyclic.



b) Since $\sphericalangle AXS = \sphericalangle TAS$ it follows easily that $\Delta AXS \sim \Delta TAS$, whence $SA^2 = ST.SX$. We have from a) that $\sphericalangle CIT = \sphericalangle CYT = \sphericalangle TXY$ and therefore $\Delta SXI \sim \Delta SIT$, whence $SI^2 = ST.SX$. Hence $SA = SI$. On the other hand, it follows from

$$\sphericalangle BCI = 180^\circ - \sphericalangle BCS = 180^\circ - \left(\gamma + \frac{\alpha+\beta}{2}\right) = 90^\circ - \frac{\gamma}{2}$$

that CI is the external bisector of $\sphericalangle ACB$.

In the isosceles ΔBSI we have $\sphericalangle BSI = \sphericalangle BSC = \alpha$ and we find $\sphericalangle BIS = 90^\circ - \frac{\alpha}{2}$. Now from ΔBCI we have $\sphericalangle CBI = 90^\circ - \frac{\beta}{2}$, which means that BI is the external bisector of $\sphericalangle ABC$. Therefore I is the center of the excircle of ΔABC tangent to the side BC.

.**3.** Assume, for a contradiction, that there exists such a set.

We first prove that if $a \in A$, then $A \cap \left(\frac{a}{2}, a\right) = \phi$. To do this suppose the contrary, i.e. there exists $a' \in A$ and $a > a' > \frac{a}{2}$. Then the number $a - a' < \frac{a}{2}$ can be represented as a sum of one or finitely many different numbers from A. Since each of these numbers is less than $\frac{a}{2}$, the number $a = a' + a - a'$ has two different representations of the required type (as a and as a' plus the numbers of the representation of $a - a'$), a contradiction.

In particular, it follows from the above that in every interval $\left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right)$, $i = 1, 2, \ldots$, there is at most one element of A. Since the set A is infinite

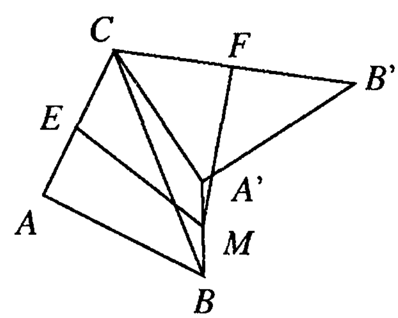
(otherwise we can obtain only a finite number of sums of different numbers from A) it easily follows that the numbers of A can be ordered in an infinite sequence $a_1, a_2, \ldots$, which satisfies $a_i \geq 2a_{i+1}$ for every i. If this inequality is strict for some i, then

$$s = \sum_{i=2}^{\infty} a_i < \sum_{i=2}^{\infty} \frac{a_1}{2^{i-1}} = a_1.$$

This shows that the numbers from the interval (s, a_1) can not be represented as a sum of one or finitely many different numbers from A.

Therefore $a_{i+1} = \dfrac{a_1}{2^i}$ for every i. Now it is easy to see that only the numbers of the form $a_1 \dfrac{m}{2^n}$ can be represented as a sum of one or a finite number of different numbers from A. Thus any rational number with an odd denominator which is coprime with the denominator of a_1 can not be represented as required. This is a contradiction which completes the proof.

4. Set $BC = B'C = a$, $AC = A'C = b$, $AB = A'B' = c$ and $\sphericalangle C = \gamma$. Since the triangles $AA'C$ and $BB'C$ are similar we have



$$\frac{AA'}{AC} = \frac{BB'}{BC} = k.$$

Therefore $AA' = kb$ and $BB' = ka$.

Using the median formula several times we get

$$4EM^2 = k^2b^2 + c^2 + b^2 + a^2 - A'B^2 - b^2,$$

$$4FM^2 = k^2a^2 + c^2 + b^2 + a^2 - A'B^2 - a^2.$$

Hence the condition $EM = FM$ is equivalent to $(k^2 - 1)(a^2 - b^2) = 0$, whence $k = 1$ (since $a \neq b$). This is true exactly when $\triangle AA'C$ is equilateral, i.e. for $\pm 60°$ rotation. We consider here only the case of a positively oriented $\triangle ABC$ and rotation through $60°$; the remaining cases are analogous. We have from $\triangle CBA'$ that

$$A'B^2 = a^2 + b^2 - 2ab\cos(60° - \gamma),$$

and we obtain from $\triangle AB'C$ that

$$4EF^2 = B'A^2 = a^2 + b^2 - 2ab\cos(60° + \gamma).$$

We shall prove that $EF = EM$. It follows from the above expressions for EF and EM that this is equivalent to the identity

$$c^2 + 2ab\cos(60° - \gamma) = a^2 + b^2 - 2ab\cos(60° + \gamma).$$

Since $a^2 + b^2 - c^2 = 2ab\cos\gamma$ we rewrite the above identity as

$$2ab\cos(60^\circ - \gamma) + 2ab\cos(60^\circ + \gamma) = 2ab\cos\gamma.$$

This is equivalent to $2\cos 60^\circ \cos\gamma = \cos\gamma$, which is true. Therefore $EF = EM$ and $\sphericalangle EMF = 60^\circ$.

5. We first prove the following lemma.

LEMMA. *If in the $(t; a, b)$-game some of the two players has a winning strategy, then in the $(t{+}a{+}b, a, b)$-game the same player has a winning strategy.*

Proof of the lemma. Denote the players by A and B and let B have a winning strategy for the $(t; a, b)$-game. In the $(t + a + b; a, b)$-game after the first move of A we obtain either the $(t + a; a, b)$-game or the $(t + b; a, b)$-game with B to go first. In both cases B can get the $(t; a, b)$-game with A as first player in which case B has a winning strategy.

Let us now assume that A has a winning strategy for the $(t; a, b)$-game. Then after the first move of A we obtain either the $(t - a; a, b)$-game or the $(t - b; a, b)$-game with B to go first. Therefore the second player has a winning strategy for some of these games.

We consider (without loss of generality) the case of the $(t - a; a, b)$-game. It follows from the above that the second player has a winning strategy for the $(t - a + a + b = t + b; a, b)$-game. Since A can obtain the $(t + b; a, b)$-game with B to go first from the $(t + a + b; a, b)$-game, it follows that A has a winning strategy for the $(t + a + b; a, b)$-game. This completes the proof of the lemma.

We now prove that for $t = 2004$ and $a + b = 2005$ the first player A has a winning strategy. We may assume that $a \le b$. Since $a > 0$, we have $b \le 2004$. Then A subtracts b from $t = 2004$. The resulting number $2004 - b$ is less than a since $a + b = 2005$. This means that any move of B leads to a negative number. Now the lemma implies that A has a winning strategy for the (t, a, b)-game for every $t \equiv 2004 \pmod{2005}$ and $a + b = 2005$.

6. We shall use the following lemma.

LEMMA. *Let x, y and n be positive integers such that $\dfrac{xy}{x+y} > n$. Then*

$$\frac{xy}{x+y} \ge n + \frac{1}{n^2 + 2n + 2}$$

with equality if and only if $\{x, y\} = \{n + 1, n^2 + n + 1\}$.

Proof of the lemma. Since $xy > n(x + y)$, we have $xy = n(x + y) + r$, where r is a positive integer. Then $(x - n)(y - n) = n^2 + r$, which implies that $x > n$ and $y > n$. Set $x = n + d_1$ and $y = n + d_2$. Then $d_1 d_2 = n^2 + r$. Now using the inequalities $\dfrac{r}{A + r} \ge \dfrac{1}{A + 1}$ and $d_1 + d_2 \le 1 + n^2 + r$ (the latter follows from

$d_1 + d_2 \leq 1 + d_1 d_2$), we get

$$\begin{aligned}\frac{xy}{x+y} &= \frac{n^2 + d_1 d_2 + n(d_1 + d_2)}{2n + d_1 + d_2} = n + \frac{r}{2n + d_1 + d_2} \\ &\geq n + \frac{r}{2n + n^2 + 1 + r} \geq n + \frac{1}{n^2 + 2n + 2}.\end{aligned}$$

Note that the equality is attained if and only if $\{x, y\} = \{n+1, n^2+n+1\}$. This completes the proof of the lemma.

The condition of the problem implies that $c(c^2 - c + 1) = pab$ and $a + b = q(c^2 + 1)$, where p and q are positive integers. Therefore

$$\frac{c(c^2 - c + 1)}{c^2 + 1} = \frac{pqab}{a+b} = \frac{xy}{x+y},$$

where $x = pqa$ and $y = pqb$. Then

$$\frac{xy}{x+y} = c - \frac{c^2}{c^2+1} = c - 1 + \frac{1}{c^2+1},$$

whence $\dfrac{xy}{x+y} > c - 1$. Now the lemma gives

$$\frac{xy}{x+y} \geq c - 1 + \frac{1}{(c-1)^2 + 2(c-1) + 2} = c - 1 + \frac{1}{c^2+1}.$$

Hence we have the case of equality and therefore

$$\{x, y\} = \{c, c^2 - c + 1\}.$$

Since the numbers c and $c^2 - c + 1$ are coprime and $x = pqa$, $y = pqb$, it follows that $p = q = 1$. Hence $\{a, b\} = \{c, c^2 - c + 1\}$.

Team selection test for 22. BMO

1. Using twice the inequalities $x - 1 < [x] \le x$ we obtain

$$abn - a - 1 < n - 1 \le abn.$$

This implies $n(ab - 1) < a$ and $n(1 - ab) \le 1$. If $ab \ne 1$, then either $ab - 1 > 0$ or $1 - ab > 0$, which shows that one of the inequalities $n(ab - 1) < a$ and $n(1 - ab) \le 1$ is not satisfied for n large enough.

Therefore $ab = 1$ and the given inequality is equivalent to

$$bn - b \le [bn] < bn.$$

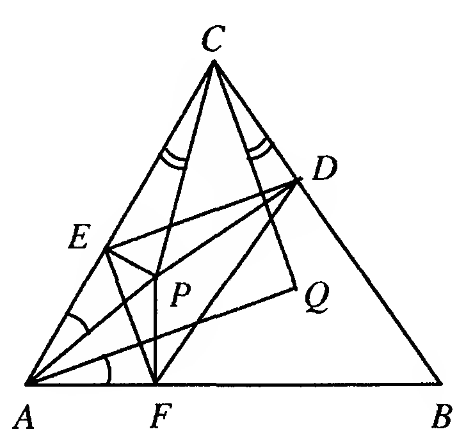
It follows from $[bn] < bn$ than bn is not integral for any n which is possible only when the number b is irrational. For $b > 1$ the inequality $bn - b \le [bn]$ is obvious. If $0 < b < 1$, we take $n = \left[\dfrac{1}{b-1}\right]$ and obtain the inequalities

$$\frac{n-2}{n-1} < b < \frac{n}{n-1}.$$

Hence $[bn] \le n - 2$ and therefore $\left[\dfrac{[bn]}{b}\right] < n - 1$, a contradiction.

Thus the solutions of the problem are the irrational numbers a and b such that $ab = 1$ and $b > 1$.

2. We have $\sphericalangle BCQ = \sphericalangle ACP = \sphericalangle EDP$. Since $PD \perp BC$, it follows that $ED \perp CQ$. Analogously we have $AQ \perp EF$. Since $\sphericalangle DEF = 90°$, we conclude that $\sphericalangle AQC = 90°$ as well. Then $\triangle QCD \sim \triangle ACP$, because $\sphericalangle QCD = \sphericalangle ACP$ and we get



$$\frac{DC}{QC} = \frac{PC\cos\sphericalangle PCD}{AC\cos\sphericalangle ACQ} = \frac{PC}{AC}.$$

Therefore $\sphericalangle DQC = \sphericalangle PAC = \sphericalangle PFE$. Since $CQ \| EF$ $(\perp ED)$, it follows that $DQ \| PF$, i.e. $DQ \perp AB$.

Using the same arguments we prove that $FQ \perp BC$ and hence Q is the orthocenter of $\triangle BDF$.

Remark. The converse assertion is also true: if Q is the orthocenter of $\triangle BDF$, then $\sphericalangle DEF = 90°$.

3. Answer: there is such a sequence. We shall define the required sequence inductively. We set $a_1 = 1$, $a_2 = 2$ and assume that $a_1, a_2, \dots, a_{2k}$ are already

determined. Denote by m the smallest positive integer which can not be represented as $a_j - a_i$, $1 \leq i < j \leq 2k$. Since the number of such differences is $d = k(2k-1)$, we have $m \leq d+1$.

Set $a_{2k+2} = a_{2k+1} + m$, where a_{2k+1} is such that

$$a_{2k+1} \neq a_l, \ a_{2k+1} \pm m \neq a_l,$$
$$a_{2k+1} - a_l \neq a_j - a_j, \ a_{2k+1} + m - a_l \neq a_j - a_j,$$

for $1 \leq l \leq 2k$, $1 \leq i < j \leq 2k$. This implies that $a_1, a_2, \ldots, a_{2k+2}$ are all distinct and every integer between 1 and m can be written in a unique way as $a_j - a_i$, $1 \leq i < j \leq 2k+2$. Since there are exactly $6k + 4kd$ "forbidden" values for a_{2k+1} we can choose a_{2k+1} with the above properties and such that $a_{2k+1} \leq 6k + 4kd + 1$. Then

$$a_{2k+1} < a_{2k+2} = a_{2k+1} + m \leq 6k + 4kd + 1 + d + 1 < (2k+1)^3$$

and it remains to put the numbers $a_1, a_2, \ldots, a_{2k+2}$ in increasing order (check that the inequality $a_n \leq n^3$ is still valid).

4. Let O be an arbitrary point in the plane and let $A_{1,1}A_{2,1}\ldots A_{n,1}$ be an n-gon similar to $\mathcal{P}$ and containing O. Consider the n-gons

$$OA_{1,1}A_{1,2}\ldots A_{1,n-1}, \ OA_{2,1}A_{2,2}\ldots A_{2,n-1}, \ldots, \ OA_{n,1}A_{n,2}\ldots A_{n,n-1},$$

that are similar to $A_{1,1}A_{2,1}\ldots A_{n,1}$ and have the same orientation. Using a rotation and a homothety with center O we see that the n-gon $A_{1,j}A_{2,j}\ldots A_{n,j}$, $j = 2, \ldots, n-1$, is similar to $A_{1,1}A_{2,1}\ldots A_{n,1}$ and has the same orientation.

Denote by o and $a_{i,j}$ the numbers assigned to the points O and $A_{i,j}$, respectively. Summing up the equalities

$$o + \sum_{j=1}^{n-1} a_{i,j} = 0, \ i = 1, \ldots, n,$$

and using that

$$\sum_{i=1}^{n} a_{i,j} = 0, \ i = 1, \ldots, n-1,$$

we get $no = 0$, which implies the assertion.

5. We shall prove by induction on n that $a_n = n - t_2(n)$, where $t_2(n)$ is the number of 1's in the binary representation of n. For $n = 0$ we have $a_0 = 0$ and the assertion is true. Let us assume that it is true for every $n \leq k-1$. If $k = 2k_0$, then $t_2(k_0) = t_2(k)$ (the binary representations of k_0 and $k = 2k_0$ have the same number of 1's) and we obtain

$$a_k = a_{k_0} + k_0 = k_0 - t_2(k_0) + k_0 = k - t_2(k_0) = k - t_2(k).$$

If $k = 2k_0 + 1$, then $t_2(2k_0 + 1) = t_2(2k_0) + 1$ (the binary representation of $2k_0 + 1$ has one more 1 than this of $2k_0$) and we get

$$\begin{aligned} a_k &= a_{k_0} + k_0 = k_0 - t_2(k_0) + k_0 = 2k_0 - t_2(2k_0) \\ &= 2k_0 + 1 - t_2(2k_0 + 1) = k - t_2(k). \end{aligned}$$

Since $n \geq 2^t - 1$ for $t_2(n) = t$, we have

$$0 \leq \lim_{n\to+\infty} \frac{t_2(n)}{n} \leq \lim_{t\to+\infty} \frac{t}{2^t} = 0.$$

Hence $\lim\limits_{n\to+\infty} \dfrac{t_2(n)}{n} = 0$ and therefore

$$\lim_{n\to+\infty} \frac{a_n}{n} = \lim_{n\to+\infty} \frac{n - t_2(n)}{n} = 1 - \lim_{n\to+\infty} \frac{t_2(n)}{n} = 1.$$

6. We shall prove the assertion by induction on $N = a_1 + a_2 + \cdots + a_m$. For $N = 1$ we have $m = 1$, $a_1 = 1$ and $b_1 = 1$ is the required number. Let us assume that the assertion is true for every collection with sum less than N and let $a_1, a_2, \ldots, a_m$ be such that $a_1 + a_2 + \cdots + a_m = N$. If all numbers $a_1, a_2, \ldots, a_m$ are even then the numbers $\dfrac{a_1}{2}, \dfrac{a_2}{2}, \ldots, \dfrac{a_m}{2}$ have sum $\dfrac{N}{2}$ and by the induction hypothesis there exists a collection $b_1, b_2, \ldots, b_n$ which satisfies the condition. Then the required numbers for $a_1, a_2, \ldots, a_m$ are $2b_1, 2b_2, \ldots, 2b_n$.

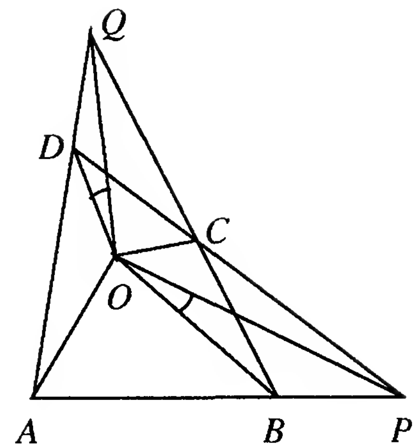
Suppose now that at least one of the numbers $a_1, a_2, \ldots, a_m$ is odd. Without loss of generality we can assume that a_m is the smallest odd number in the collection. Let us consider the numbers $a_1', a_2', \ldots, a_{m-1}'$ defined by

$$a_i' = \begin{cases} \dfrac{a_i}{2} & \text{, if } a_i \text{ is even} \\ \dfrac{a_i - a_m}{2} & \text{, if } a_i \text{ is odd} \end{cases}$$

The sum of the new numbers a_i' is less than N and the induction hypothesis implies the existence of numbers $b_1', b_2', \ldots, b_k'$ which satisfy the conditions. We shall prove that the numbers $2b_1', 2b_2', \ldots, 2b_k', a_m$ are the required numbers for the collection $a_1, a_2, \ldots, a_m$.

If two nonintersecting subsets of $\{2b_1', 2b_2', \ldots, 2b_k', a_m\}$ have equal sums then a_m (as the only odd number) does not belong to these sets. Dividing by 2 we obtain two nonintersecting subsets of $\{b_1', b_2', \ldots, b_k'\}$ with equal sums which is a contradiction. Also, it is easy to see that every a_i, $i = 1, 2, \ldots, m$ can be represented as a sum of some of the numbers $2b_1', 2b_2', \ldots, 2b_k', a_m$ which completes the induction step.

7. The Sine theorem for the triangles ODQ and AOQ gives



$$\frac{\sin\varphi}{\sin\beta} = \frac{QD}{OD}$$

and

$$\frac{\sin(\varphi + \sphericalangle AOD)}{\sin\beta} = \frac{AQ}{OA},$$

whence

$$\frac{\sin(\varphi + \sphericalangle AOD)}{\sin\varphi} = \frac{AQ}{OA}\cdot\frac{OD}{QD}.$$

We obtain in the same way that

$$\frac{\sin(\varphi + \sphericalangle AOB)}{\sin\varphi} = \frac{AP}{OA}\cdot\frac{OB}{BP}.$$

Therefore

$$\frac{\sin(\varphi + \sphericalangle AOD)}{\sin(\varphi + \sphericalangle AOB)} = \frac{AQ}{AP}\cdot\frac{OD}{OB}\cdot\frac{BP}{QD}.$$

We get in the same way that

$$\frac{\sin(\sphericalangle DOC - \varphi)}{\sin(\sphericalangle BOC - \varphi)} = \frac{QC}{PC}\cdot\frac{OB}{OD}\cdot\frac{PD}{QB}.$$

Using the Menelaus theorem for $\triangle ADC$ and the line QP and for $\triangle ABC$ and the line QP we obtain

$$\frac{AQ.DP}{DQ.CP} = \frac{AL}{CL} \quad\text{and}\quad \frac{QC.BP}{QB.AP} = \frac{CL}{AL}.$$

Setting $\varphi + \sphericalangle AOD = x$, $\varphi + \sphericalangle AOB = y$, $\sphericalangle DOC - \varphi = z$ and $\sphericalangle BOC - \varphi = t$, we have

$$\frac{\sin x.\sin z}{\sin y.\sin t} = \frac{AQ.DP.QC.BP}{DQ.CP.QB.AP} = \frac{AL}{CL}\cdot\frac{CL}{AL} = 1,$$

i.e. $\sin x.\sin z = \sin y.\sin t$. It follows easily from here that

$$\cos(x - z) - \cos(x + z) = \cos(y - t) - \cos(y + t).$$

Since $x + y + z + t = 360^\circ$, we have $\cos(x + z) = \cos(y + t)$ and therefore $\cos(x - z) = \cos(y - t)$. Since $x - z + y - t < 360^\circ$ and the equality $x - z = t - y$ implies that O lies on PQ (prove this!) we obtain $x - z = y - t$, whence $x + t = z + y = 180^\circ$.

8. If for every $s = 1, 2, \ldots, B$ any s boys know together at least s girls then the Hall (marriages') theorem implies that every boy can dance with a known

girl and the condition is satisfied. Let us assume now the converse and choose the largest $s \leq B$, such that there are s boys who know together at most $s-1$ girls.

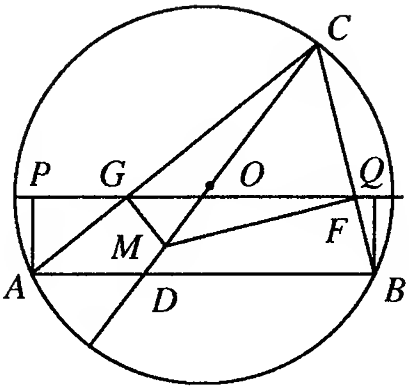
Denote the set of these s boys by S and let L be the set of girls known to the boys from S. If some t of the boys outside S know together at most t of the girls outside L we have a contradiction with the choice of s. Therefore every t boys outside S know together at least $t+1$ of the girls outside L. Now the Hall theorem implies that every boy outside S can dance with a known girl outside L. Hence still non-dancing girls outside L are at least

$$G-(B-s)-(s-1)=G+1-B \geq B$$

(the girls which dance with boys outside S are $B-s$ and the girls which are known to the boys from S are at most $s-1$). If the boys from S dance with some s of these remaining non-dancing girls outside L then the condition is satisfied.

Team selection test for 46. IMO

1. Let $AP \perp FG$ and $BQ \perp FG$, $P, Q \in FG$. Then



$$AB \geq PQ = PG + GF + FQ.$$

Since the quadrilateral $CFMG$ is cyclic we obtain $\sphericalangle CMF = \sphericalangle CGF = \sphericalangle AGP$. This implies that $\triangle APG \sim \triangle CFM$ and therefore

$$\frac{PG}{AG} = \frac{MF}{CM} \iff PG = \frac{MF.AG}{CM}.$$

In the same way we have $QF = \dfrac{MG.BF}{CM}$. Then

$$AB - FG \geq PG + FQ = \frac{MF.AG + MG.BF}{CM}.$$

It is clear that equality is attained iff $AB \| FG$. The last means that $\sphericalangle BAC = \sphericalangle FGC = \sphericalangle AGP$, whence $\sphericalangle MCB = 90° - \sphericalangle BAC = \sphericalangle OCB$, where O is the circumcenter of $\triangle ABC$. Therefore the required locus is the segment CD, where D is the intersection point of the line OC and the side AB.

2. Let us consider the set $\{1, 2, 2^2, \dots, 2^{10}\}$. Since every number from 0 to 2047 can be represented in a unique way as a sum of powers of 2 (elements of our set), we conclude that for every i, $0 \leq i \leq 2047$, there is a unique subset of $\{1, 2, 2^2, \dots, 2^{10}\}$ such that the sum of its elements is equal to i (0 corresponds to the empty set).

We now consider a set A with the following property: for every i the number of the subsets of A such that the sums of their elements are congruent to i modulo 2048 does not depend on i. It is easy to see that for every a the set $A \cup \{a\}$ has the same property. Since $\{1, 2, 2^2, \dots, 2^{10}\} \subset \{1, 2, 3, \dots, 2005\}$, we conclude that the number of the subsets B of $\{1, 2, 3, \dots, 2005\}$ such that the sums of their elements are congruent to i modulo 2048 is equal to $\dfrac{2^{2005}}{2048} = \dfrac{2^{2005}}{2^{11}} = 2^{1994}$ and this is the required number.

3. Set $\alpha = f(1)$. Then setting $y = 1$ and $x = 1$ in

$$f(x^2 + y) = f^2(x) + \frac{f(xy)}{f(x)} \tag{1}$$

gives

$$f(x^2 + 1) = f^2(x) + 1 \tag{2}$$

and

$$(3) \qquad f(y+1) = \alpha^2 + \frac{f(y)}{\alpha},$$

respectively. Using (3), we consecutively get

$$f(2) = \alpha^2 + 1,\ f(3) = \frac{\alpha^3 + \alpha^2 + 1}{\alpha},$$

$$f(4) = \frac{\alpha^4 + \alpha^3 + \alpha^2 + 1}{\alpha^2},\ f(5) = \frac{\alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + 1}{\alpha^3}.$$

On the other hand, setting $x = 2$ in (2) gives $f(5) = \alpha^4 + 2\alpha^2 + 2$. Therefore

$$\frac{\alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + 1}{\alpha^3} = \alpha^4 + 2\alpha^2 + 2 \iff \alpha^7 + \alpha^5 - \alpha^4 + \alpha^3 - \alpha^2 - 1 = 0,$$

whence

$$(\alpha - 1)\left[\alpha^4(\alpha^2 + \alpha + 1) + (\alpha + 1)^2(\alpha^2 - \alpha + 1) + 2\alpha^2\right] = 0.$$

Since the expression in the square brackets is positive, we have $\alpha = 1$. Now (3) implies that

$$(4) \qquad f(y+1) = f(y) + 1$$

and therefore $f(n) = n$ for every positive integer n.

Now take an arbitrary positive rational number $\frac{a}{b}$ (a, b are positive integers). Since (4) gives $f(y) = y \iff f(y+m) = y+m$, m is a positive integer, the equality $f\left(\frac{a}{b}\right) = \frac{a}{b}$ is equivalent to

$$f\left(b^2 + \frac{a}{b}\right) = b^2 + \frac{a}{b}.$$

Since the last equality follows from (1) for $x = b$ and $y = \frac{a}{b}$ we conclude that $f\left(\frac{a}{b}\right) = \frac{a}{b}$.

Setting $y = x^2$ in (4), we obtain $f(x^2 + 1) = f(x^2) + 1$. Hence using (2) we conclude that $f(x^2) = f^2(x) > 0$. Thus $f(x) > 0$ for every $x > 0$. Now (1), the inequality $f(x) > 0$ for $x > 0$ and the identity $f(x^2) = f^2(x)$ imply that $f(x) > f(y)$ for $x > y > 0$. Since $f(x) = x$ for every rational number $x > 0$, it easily follows that $f(x) = x$ for every real number $x > 0$.

Finally, given an $x < 0$ we choose $y < 0$ such that $x^2 + y > 0$. Then $xy > 0$ and (1) gives

$$\begin{aligned} x^2 + y &= f(x^2 + y) = f^2(x) + \frac{f(xy)}{f(x)} \\ &= f(x^2) + \frac{xy}{f(x)} = x^2 + \frac{xy}{f(x)}, \end{aligned}$$

i.e. $f(x) = x$. Therefore $f(x) = x$ for every $x \in \mathbb{R}^*$. It is clear that this function satisfies (1).

4. We first prove that at least one of the numbers $a_1, a_2, \ldots, a_{2005}$ is not positive. To do this we assume the contrary and choose i such that

$$\frac{b_i}{a_i} = M = \max_{1 \le j \le 2005} \left(\frac{b_j}{a_j}\right).$$

Then we can find $\varepsilon > 0$ such that

$$(a_i x - b_i)^2 < \sum_{j=1, j \neq i}^{2005} (a_j x - b_j)$$

for every $x \in (M, M + \varepsilon)$, a contradiction.

On the other hand, it is easy to see that if $a_1 = a_2 = \cdots = a_{2004} = -a_{2005} = 1$ and $b_1 = b_2 = \cdots = b_{2004} = b_{2005} \geq \dfrac{1001^2}{2}$ the given inequality is satisfied. Therefore the answer is 4009.

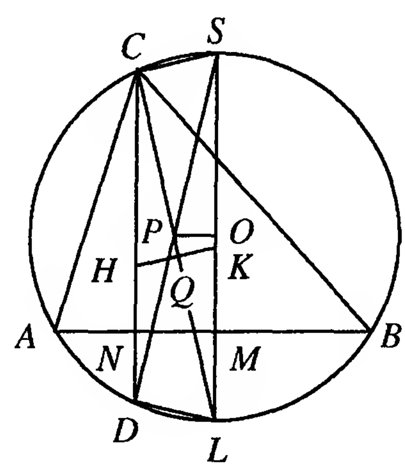
5. Denote by $k(O, R)$ the circumcircle of $\triangle ABC$ and by Q the orthogonal projection of H on CL. Let us further denote

$$K = HQ \cap LO,\ S = k \cap LO,\ P = CL \cap DS,$$

$$M = AB \cap LO,\ N = AB \cap CD.$$

Note that N and M are the midpoints of HD and KL, respectively. Indeed, we have $AH = AD$ ($\sphericalangle AHD = \sphericalangle ABC = \sphericalangle ADH$) and analogously $BH = BD$. Hence AB is the perpendicular bisector of the segment HD.

On the other hand, $DLSC$ is an isosceles trapezoid and $HK || CS$. Therefore $DLKH$ is also an isosceles trapezoid and AB is the perpendicular bisector of the segment KL. Then



$$\frac{LQ}{LC} = \frac{LK}{LS} = \frac{LM}{R},$$

whence

$$LQ = \frac{LC.LM}{R}.$$

We also have $LP = \dfrac{LO.LS}{LC} = \dfrac{2R^2}{LC}$ since $\triangle LOP \sim \triangle LCS$ and $LB^2 = LS.LM = 2R.LM$ since $\triangle LBS \sim \triangle LMB$. Therefore

$$LP.LQ = LB^2.$$

On the other hand, using $\sphericalangle LBI = \frac{\sphericalangle B + \sphericalangle C}{2} = \sphericalangle LIB$ we obtain $LB = LI$. Then $LP.LQ = LI^2$ and, in particular, $Q \equiv I \iff P \equiv I$.

It remains to note that $\sphericalangle CIH = 90^\circ \iff Q \equiv I$ (since $\sphericalangle CQH = 90^\circ$) and $\sphericalangle IDL = 90^\circ \iff P \equiv I$ (since $\sphericalangle PDL = 90^\circ$). This completes the proof.

Remark. It can be proved that $\sphericalangle CIH = 90^\circ$ iff $\cos \sphericalangle A + \cos \sphericalangle B = 1$.

6. We have to prove that if the edges of a complete graph with 9 vertices are colored in blue and red in such a way that there is no blue quadrilateral then its vertices can be partitioned into 4 groups without any blue edges inside any group.

LEMMA 1. *The edges of a complete graph with 6 vertices are colored in blue and red in such a way that there is no blue quadrilateral and its vertices can not be partitioned into 3 groups without blue edges inside any group. Then the graph does not contain a red triangle.*

Proof. It is well known that the complete graph with 6 vertices and edges colored in two colors (red and blue) has an one-colored triangle. Denote the vertices of the graph by $v_1, v_2, \dots, v_6$ and assume the existence of a red triangle $v_1v_2v_3$. If some edge v_iv_j, $i, j \in \{4, 5, 6\}$, is red, then $\{v_1, v_2, v_3\}, \{v_i, v_j\}$ and $\{v_k\}$, $k \neq 1, 2, 3, i, j$, is a partition which is supposed not to exist.

Therefore $v_4v_5v_6$ is a blue triangle. Now the condition implies that for every $i = 1, 2, 3$ at least one of the edges v_iv_j, $j = 4, 5, 6$, is red. If the edges v_1v_4 and v_2v_4 are red then $v_1v_2v_4$ is a red triangle and, as above, $v_3v_5v_6$ is a blue triangle. Now, if the edge v_3v_4 is blue, then $v_3v_4v_5v_6$ is a blue quadrilateral and, if the edge v_3v_4 is red, then we have the partition $\{v_1, v_2, v_3, v_4\}$, $\{v_5\}$ and $\{v_6\}$, a contradiction, which completes the proof of the lemma.

LEMMA 2. *Under the conditions of Lemma 1 all red edges of G form a cycle of length 5.*

Proof. We know from Lemma 1 that G does not contain a red triangle.

Without loss of generality we can assume that the edges v_1v_2 and v_3v_4 are red. If all edges v_iv_j for $i = 1, 2$, $j = 3, 4$ are red, then the partition $\{v_1, v_2, v_3, v_4\}$, $\{v_5\}$ and $\{v_6\}$ ensures a contradiction. So, let v_2v_4 be blue. Since the quadrilateral $v_2v_4v_5v_6$ can not be blue, we may assume without loss of generality that the edge v_4v_6 is red. Now v_3v_6 is blue (otherwise $v_3v_4v_6$ is a red triangle) and v_3v_5 is blue (otherwise we have the partition $\{v_1, v_2\}$, $\{v_3, v_5\}$ and $\{v_4, v_6\}$ for a contradiction).

If the edge v_1v_3 is red then using the conditions of the problem we see that the edge v_2v_5 is blue, the edge v_2v_6 is red and the edges v_1v_6, v_1v_5, v_4v_5 and v_1v_4 are blue. We finally conclude that the red edges in G are v_1v_2, v_2v_6, v_6v_4, v_4v_3 and v_3v_1 and they form a cycle of length 5 as desired.

If the edge v_1v_3 is blue, then similar arguments lead to a contradiction. This completes the proof of the lemma.

Let us now consider a graph with 9 vertices $v_1, v_2, \ldots, v_9$, satisfying the given conditions. It is known that such a graph contains a red triangle or a blue quadrilateral. Since the latter is impossible we have a red triangle $v_7v_8v_9$, say. If the induced graph $v_1 \ldots v_6$ can be divided into three groups without blue edges inside any group then we obtain the required division of G.

Otherwise Lemma 2 implies that we may assume without loss of generality that the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5 and v_5v_1 are red.

If the edges v_iv_6, $i = 7, 8, 9$, are red then $\{v_6, v_7, v_8, v_9\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$ and $\{v_5\}$ is the desired division. If the edge v_7v_6 is blue while the edges v_8v_6 and v_9v_6 are red then at least three of the edges v_7v_i, $i = 1, 2, \ldots, 5$, are red and we easily have the desired division (otherwise we get a red quadrilateral).

Similar arguments solve the two remaining cases: when two of the edges v_6v_7, v_6v_8 and v_6v_9 are blue and one is red, and when all of them are blue.

Bulgarian Mathematical Competitions 2006

Winter Mathematical Competition

9.1. If x_0 is a common root of the equations, then

$$x_0(a^2-b^2)=a^3-b^3.$$

Case 1. For $a\neq b$ we have $x_0=\dfrac{a^2+ab+b^2}{a+b}$. Since $x_0>0$, it follows by the first equation that $x_0^2+a^2x_0+b^3>0$, a contradiction.

Case 2. If $a=b$, then the equations coincide. They have a real root when $D=a^4-4a^3=a^3(a-4)\geq 0$. Since $a\geq 0$, we conclude that the solutions of the problem are the pairs (a,a), where $a\in\{0\}\cup[4,+\infty)$.

9.2. It follows from the condition $x_1=x_2^2+x_2$ and Vieta's formulas that

$$\left|\begin{array}{rcr} x_1+(b-1)x_2 & = & -c \\ x_1+x_2 & = & -b \\ x_1x_2 & = & c \end{array}\right. .$$

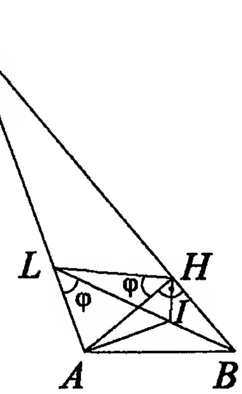
Hence $c^2+4(1-b)c+b^3-b^2=0$, $b\neq 2$.

a) Since $c=4-b$, we obtain $b^3+4b^2-28b+32=0$ which is equivalent to $(b-2)^2(b+8)=0$. Therefore $b=-8$ and $(b,c)=(-8,12)$.

b) Consider $c^2+4(1-b)c+b^3-b^2=0$ as a quadratic equation of c. It follows that $D=16(1-b)^2-4(b^3-b^2)=4(1-b)(b-2)^2$ is a perfect square. Thus $b=2$ or $1-b=k^2$, where k is an integer. Then $(b,c)=(2,2)$ or $(b,c)=(1-k^2,k(k-1)^2)$. Obviously the first pair is not a solution of the problem. The integers in the second pair are coprime when $k-1=\pm1$, that is, $k=2$ or $k=0$. So $(b,c)=(-3,2)$ or $(b,c)=(1,0)$. In both cases the roots of the given equation are real and distinct.

9.3. ($\Rightarrow$) Let $\sphericalangle AHL=\sphericalangle ALB=\varphi$. Denote by I the incenter of $\triangle ABH$. Then $\sphericalangle AHI=\frac{1}{2}\sphericalangle AHB=45^\circ$ and $\sphericalangle AIL=180^\circ-\sphericalangle AIB=45^\circ$. Hence $\sphericalangle LAI+\sphericalangle LHI=(180^\circ-\sphericalangle ALI-\sphericalangle AIL)+(\sphericalangle AHL+\sphericalangle AHI)=(180^\circ-\varphi-45^\circ)+(\varphi+45^\circ)=180^\circ$.

It follows that the quadrilateral $AIHL$ is cyclic. Then $\varphi=45^\circ$ and



$$\sphericalangle BAC=90^\circ+\sphericalangle BAI=90^\circ+\frac{1}{2}(90^\circ-\sphericalangle ABC).$$

Since $\sphericalangle ABC=180^\circ-\sphericalangle BAC-\sphericalangle ACB$, we conclude that

$$\sphericalangle BAC=\sphericalangle ACB+90^\circ.$$

(⇐) Let $\sphericalangle BAC = 90^\circ + \sphericalangle ACB$. Then AL is the external bisector of $\sphericalangle BAH$. Hence L is the center of the excircle of $\triangle ABH$ tangent to the side AH and $\sphericalangle AHL = \frac{1}{2}\sphericalangle CHA = 45^\circ$.

On the other hand,

$$\begin{aligned}\sphericalangle ALB &= 180^\circ - \sphericalangle BAL - \sphericalangle ABL = 180^\circ - \sphericalangle BAC - \frac{\sphericalangle ABC}{2}\\ &= 180^\circ - \sphericalangle BAC - \frac{180^\circ - \sphericalangle BAC - \sphericalangle ACB}{2}\\ &= 90^\circ - \frac{\sphericalangle BAC - \sphericalangle ACB}{2} = 45^\circ.\end{aligned}$$

It follows that $\sphericalangle AHL = \sphericalangle ALB$.

9.4. On the figure below it is shown how 37 tokens can be placed in a way to satisfy (1) and (2). Now we shall prove that 37 is the desired number.

Consider the columns of the table of size 6×6 obtained by cutting outmost rows and columns of the given table. It follows from (1) that there are at least 3 tokens in every such column. If there are 3 tokens in a column 6×1 with no neighbors we have a contradiction to (2). Therefore in a column with 3 tokens they are placed either in the second, third and fifth cell or in the second, forth and fifth cell.

	•		•		•	•	
•		•		•	•		•
	•		•	•		•	
•		•	•		•		•
	•	•		•		•	•
•	•		•		•	•	
•		•		•	•		•
	•		•	•		•	

Denote by k the number of columns with 3 tokens each. There are at least 4 tokens in each of the remaining $6 - k$ columns of a table 6×6 and the two outmost columns of the initial table. Note that by (1) there are 5 tokens in each column of the initial table with 3 tokens in the table 6×6.

Suppose that there are two neighboring columns having 3 tokens each. Then there exists a rectangle 2×1 without a token, a contradiction. Therefore there are at most 3 columns having 3 tokens each, i.e. $k \leq 3$.

Consider the two rectangles 6×1 above and under the table 6×6. There are two cases:

Case 1. There are at most 3 tokens in one of these rectangles. Now, there are at least 5 tokens in the outmost columns of the initial table and therefore there are at least

$$5k + 2 \cdot 5 + 4(6 - k) + 2(3 - k) = 40 - k \geq 37$$

tokens on the table.

Case 2. There are at least 4 tokens in both rectangles. Then the total number of tokens is at least

$$5k + 4(8 - k) + 2(4 - k) = 40 - k \geq 37.$$

Hence the desired number is 37.

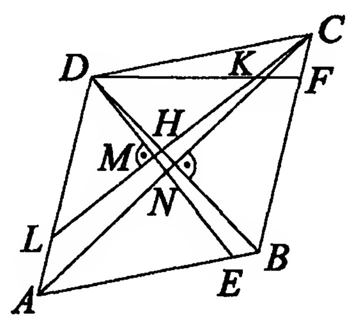
10.1. a) For $a = 3$ and $x \in [0,2]$ the inequality is equivalent to $2\sqrt{x(2-x)} \geq 1$, i.e. $4x^2 - 8x + 1 \leq 0$. Hence its solutions are

$$x \in \left[\frac{2-\sqrt{3}}{2}, \frac{2+\sqrt{3}}{2}\right].$$

b) For $a \geq 0$ and $x \in [0,2]$ the inequality is equivalent to $2\sqrt{x(2-x)} \geq a-2$. If $a \leq 2$, then any $x \in [0,2]$ is a solution and the condition of the problem does not hold.

Let $a > 2$. Then $4x(2-x) \geq (a-2)^2$ (in particular, $x \in [0,2]$), i.e. $4x^2-8x+a^2-4a+4 \leq 0$. It follows that $D = 16a(4-a) \geq 0$ and hence $a \in (2,4]$. In this case the solutions of the inequality are $x \in [x_1, x_2]$, where $x_1 \leq x_2$ are the roots of the respective quadratic equation. The given condition becomes $x_2 - x_1 \leq \sqrt{3}$. Since $x_2 - x_1 = \dfrac{\sqrt{D}}{4} = \sqrt{a(4-a)}$, we obtain $a^2 - 4a + 3 \geq 0$. Taking into account that $a \in (2,4]$ we conclude that $a \in [3,4]$.

10.2. a) Let $M = DE \cap CL$ and $K = DF \cap CL$. Then DM is an altitude and a bisector in $\triangle LKD$, hence $DL = DK$. Since $\triangle LKD \sim \triangle CKF$, it follows that $KF = CF$ and $AE = DF - CF = DF - KF = DK = DL$.



b) Since $\triangle ANE \sim \triangle CND$, $\triangle HNC \sim LAC$ and $\triangle LHD \sim \triangle CHB$, we get

$$\frac{AE}{CD} = \frac{AN}{NC} = \frac{LH}{HC} = \frac{DL}{BC}.$$

Then the equality $AE = DL$ implies that $BC = CD$.

c) It follows by b) that $ABCD$ is a rhombus. Then $DB \perp AC$ and hence H is the orthocenter of $\triangle DNC$. So $HN \perp DC$, which implies $AD \perp DC$. Thus, $ABCD$ is a square.

10.3. It follows that $2^t \equiv 1 \pmod 3$ and therefore t is even. Also $2^t \equiv 2^z \pmod 5$, i.e. $2^{t-z} \equiv 1 \pmod 5$ (obviously $t > z$). Then 4 divides $t - z$ and hence 2 divides z.

Further, it is clear that $t \geq 6 > 2$ and therefore $0 \equiv 3^x(-3)^y + (-1)^z \pmod 8$ or, equivalently, $3^{x+y} \equiv (-1)^{y+1} \pmod 8$. If y is even, then $3^{x+y} \equiv -1 \pmod 8$, a contradiction. Hence y is odd and $3^{x+y} \equiv 1 \pmod 8$.

It follows that $x+y$ is even and hence x is odd. Set $t = 2m$ $(m \geq 3)$, $z = 2n$ $(n \geq 1)$ and write the equation in the form

$$(2^m - 7^n)(2^m + 7^n) = 3^x 5^y.$$

Since $(2^m - 7^n, 2^m + 7^n) = 1$, the following three cases are possible:

Case 1. $2^m - 7^n = 3^x, 2^m + 7^n = 5^y$;
Case 2. $2^m - 7^n = 5^y, 2^m + 7^n = 3^x$;
Case 3. $2^m - 7^n = 1, 2^m + 7^n = 3^x 5^y$.

In the first two cases we have $2^m \mp 7^n = 3^x$. Having in mind that $m \geq 3$ and x is odd, we get $\mp(-1)^n \equiv 3 \pmod 8$, i.e. $3 \equiv \pm 1 \pmod 8$, a contradiction.

In the third case the equality $2^m - 7^n = 1$ implies that $2^m \equiv 1 \pmod 7$. Hence 3 divides m. Set $m = 3k$. It follows that $(2^k - 1)(2^{2k} + 2^k + 1) = 7^n$. It is easy to see that $(2^k - 1, 2^{2k} + 2^k + 1)$ equals 1 or 3. Hence $2^k - 1 = 1$, $2^{2k} + 2^k + 1 = 7^n$. Then $k = 1$, $n = 1$, $m = 3$, $t = 6$, $z = 2$ and we get $x = y = 1$.

In conclusion, the only solution of the problem is $t = 6$, $x = 1$, $y = 1$, $z = 2$.

10.4. a) There are $\frac{40.39}{2} = 20.39$ pairs of knights. Since there are 20 pairs every morning we need at least 39 days. For 39 days the arrangement of the fights can be done in the following way: place 39 of the knights A_1, A_2, ..., A_{39} at the vertices of a regular 39-gon and place the last knight B at its center.

Let B fight A_i on the day i and the remaining fights be A_{i-j} against A_{i+j} (the chord $A_{i-j}A_{i+j}$ is perpendicular to BA_i, where the indices are taken modulo 39. Since 39 is an odd number every chord is perpendicular to one radius and therefore every pair fights in a certain day.

б) The neighboring pairs are $\frac{40.39.2}{2} = 40.39$. Since there are 40 pairs we need at least 39 evenings. Using a) the needed arrangement for 39 days can be done in the following way: connect all segments corresponding to the fights on days i and $i + 1$ (the days are numbered modulo 39). We obtain the closed broken line

$$BA_iA_{i+2}A_{i-2}A_{i+4}A_{i-4}\dots A_{i+38}A_{i-38}$$

(note that $A_{i-38} = A_{i+1}$), which includes 40 points without repetition (no two indices differ by 39 because of parity arguments and since the largest difference equals $38 - (-38) < 2.39$).

Therefore this broken line contains all 40 points and we take this distribution of the knights around the table at evening i. According to a) every two knights are neighbors on the day before their fight and on the day of the fight.

11.1. It is clear that $a > 0$, $a \neq 1$. We have

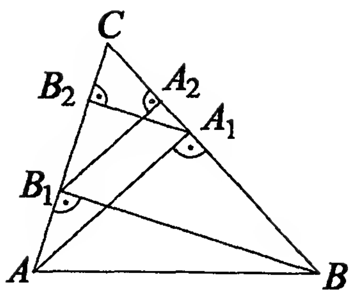
$$a^{x^2+x}(a^2+1) = a^{2(x^2+x)} + a^2.$$

Setting $u = a^{x^2+x}$ gives the equation

$$u^2 - (a^2+1)u + a^2 = 0$$

with roots 1 и a^2. Then $x^2 + x = 0$ and $x^2 + x - 2 = 0$, respectively. Thus, for any $a > 0$, $a \neq 1$, the equation has four roots $x = -2, -1, 0, 1$.

11.2. We obviously have $CA_1 = \frac{1}{2}CA_0$, $CA_2 = \frac{1}{4}CA_0$ and so on. Therefore $CA_{2006} = \frac{1}{2^{2006}}CA_0 = \frac{1}{2^{2006}}CA$ and analogously $CB_{2006} = \frac{1}{2^{2006}}CB$.



Then it follows that $A_{2006}B_{2006} \parallel AB$ and $A_{2006}B_{2006} = \frac{1}{2^{2006}}AB$. Since the line $A_{2006}B_{2006}$ is tangent to the incircle of $\triangle ABC$ if and only if the quadrilateral $ABB_{2006}A_{2006}$ is cyclic, we have

$$AB + A_{2006}B_{2006} = AA_{2006} + BB_{2006}$$
$$\Leftrightarrow AB + \frac{AB}{2^{2006}} = \frac{(2^{2006}-1)(AC+BC)}{2^{2006}}$$
$$\Leftrightarrow \frac{AC+BC}{AB} = \frac{2^{2006}+1}{2^{2006}-1}.$$

11.3. Using the formula $\cos 2\alpha = 2\cos^2\alpha - 1$, the equation becomes

$$a(\cos^2 x + \cos^2 y + \cos^2 z) + (1-a)(\cos x + \cos y + \cos z) + 3 - 6a = 0.$$

Consider the function $f(t) = at^2 + (1-a)t + 1 - 2a$, $t \in [-1,1]$. The roots of the equation $f(t) = 0$ are $t_1 = -1$ and $t_2 = \frac{2a-1}{a}$, $a \neq 0$. The following three cases are possible:

Case 1. $a < 0$. Since $\frac{2a-1}{a} > 1$, it follows that $f(t) \geq 0$ for any $t \in [-1,1]$ and $f(t) = 0$ if and only if $t = -1$.

Case 2. $a = 0$. Then $f(t) = t + 1 \geq 0$ for any $t \in [-1,1]$ and $f(t) = 0$ if and only if $t = -1$.

Case 3. $a > 0$. Then $a \geq 1$ and hence $\frac{2a-1}{a} \geq 1$ with equality for $a = 1$. It follows that $f(t) \geq 0$ for any $t \in [-1,1]$. Moreover, if $a > 1$ then $f(t) = 0$ for $t = -1$, and if $a = 1$, then $f(t) = 0$ for $t = \pm 1$.

Since the given equation has the form $f(\cos x) + f(\cos y) + f(\cos z) = 0$, we conclude that:

– if $a \neq 1$, then $\cos x = \cos y = \cos z = -1$. Hence the solutions of the problem are $x = (2k+1)\pi$, $y = (2l+1)\pi$, $z = (2m+1)\pi$, where $k, l, m \in \mathbb{Z}$.

– if $a = 1$, then in addition to the above solutions we also have $\cos x = \cos y = \cos z = 1$, that is, $x = 2r\pi$, $y = 2s\pi$, $z = 2t\pi$, where $r, s, t \in \mathbb{Z}$.

11.4. a) Let $n > 1$ and $a = \overline{a_1 a_2 \ldots a_n}$ be a bad integer with digits 1, 2 and 3. Since 3 divides exactly one of the integers $\overline{a_{n-1}a_n 1}$, $\overline{a_{n-1}a_n 2}$ and $\overline{a_{n-1}a_n 3}$, then exactly two of them are bad. It follows that adding 1, 2 or 3 to a one obtains exactly two bad integers with $n+1$ digits. Since the number of the two-digit integers whose decimal representations contain only the digits 1, 2 or 3 equals 9, the answer of a) is $9 \cdot 2^{2004}$.

b) The integers 122122...12212 and 233233...23323 are bad and their sum 355355...35535 is also a bad integer. Thus 0 is one of the possible values of k.

Let now $a = \overline{a_1a_2\dots a_n}$ and $b = \overline{b_1b_2\dots b_n}$ be different bad integers and suppose that their sum is also a bad integer. Then 3 does not divide $a_i+a_{i+1}+a_{i+2}$, $b_i+b_{i+1}+b_{i+2}$ and $a_i+a_{i+1}+a_{i+2}+b_i+b_{i+1}+b_{i+2}$. This means that $a_i+a_{i+1}+a_{i+2} \equiv b_i+b_{i+1}+b_{i+2} \equiv 1,2 \pmod 3$. Assume that two of the digits a_i, a_{i+1}, a_{i+2} coincide with the respective digits b_i, b_{i+1}, b_{i+2}. It follows from above that the third digits also coincide. Continuing in the same way, we conclude that $a = b$, a contradiction.

So, among any three consecutive digits of a, at most one coincides with the respective digit of b. On the other hand, if $a_i = b_i$, then $a_{i+3} = b_{i+3}$ (and analogously $a_{i-3} = b_{i-3}$). Indeed $a_i+a_{i+1}+a_{i+2} \equiv b_i+b_{i+1}+b_{i+2} \pmod 3$ implies that $a_{i+1}+a_{i+2} \equiv b_{i+1}+b_{i+2} \pmod 3$. If $a_{i+3} \neq b_{i+3}$, then $a_i+a_{i+1}+a_{i+2} \equiv b_i+b_{i+1}+b_{i+2} \pmod 3$ which is impossible.

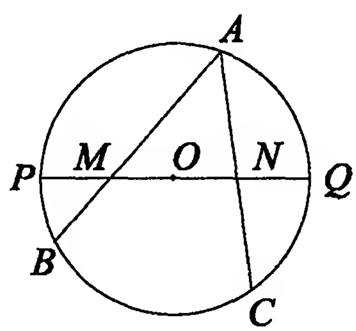
Thus, if $k > 0$, then among any three consecutive digits of a exactly one coincides with the respective digit of b. It follows that $k = 669$ or $k = 668$.

12.1. a) We have

$$f'(x) = \frac{(2x-2006)(x^2+1) - 2x(x^2-2006x+1)}{(x^2+1)^2} = \frac{2006(x^2-1)}{(x^2+1)^2}.$$

Hence $f'(x) \geq 0$ if and only if $x \in (-\infty, -1] \cup [1, +\infty)$.

b) It follows from a) that $f(x)$ increases for $x \in (-\infty, -1) \cup (1, +\infty)$ and decreases for $x \in (-1, 1)$. Hence its maximum equals $f(-1) = 1004$ and its minimum equals $f(1) = -1002$. Then $|f(x) - f(y)| \leq |1004 - (-1002)| = 2006$ for any x and y.



12.2. Let M and N lie on the diameter PQ ($M \in PO, N \in QO$) of k. Set $x = MO = NO$, $0 \leq x \leq \sqrt{5}$. Then

$$MA.MB = MP.MQ = (\sqrt{5}-x)(\sqrt{5}+x) = 5-x^2.$$

Analogously $NA.NC = 5 - x^2$. It follows that

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{MA^2+NA^2}{(5-x^2)^2}.$$

Using the median formula we get

$$5 = AO^2 = \frac{1}{4}[2(MA^2+NA^2) - 4x^2],$$

i.e. $MA^2 + NA^2 = 2(5+x^2)$. Therefore

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{2(5+x^2)}{(5-x^2)^2} = \frac{3}{4x^2}.$$

Hence $x^4 + 14x^2 - 15 = 0$, i.e. $x = 1$.

12.3. Let C be a set of phone numbers satisfying the given three conditions. Assume that C has maximal cardinality. Denote by A the set of phone numbers in C which have four or five equal digits, and by B the set of phone numbers in C which have exactly three equal digits. Obviously, $C = A \cup B$. Also $|A| \leq 10$, since any digit can appear four or five times in at most one number in C.

Denote by $B_{i,j}$, $0 \leq i, j \leq 9$, $i \neq j$, the set of phone numbers containing three digits i and two digits j. We shall prove that the maximal cardinality of $B_{i,j} \cup B_{j,i}$ is 4. It is enough to consider the case $i = 0$, $j = 1$. Let a_i be the number of phone numbers in $B_1 = B_{0,1} \cup B_{1,0}$ with i blocks (a sequence $a_k, \dots, a_j$ is called a block if $a_{k-1} \neq a_k = \cdots = a_j \neq a_{j+1}$.)

Assume that $|B_1| = 5$. Then

$$\begin{aligned} a_2 + a_3 + a_4 + a_5 &= 5, \\ 2a_2 + 3a_3 + 4a_4 + 5a_5 &\leq 14 \end{aligned}$$

since any two phone numbers have no common subsequence of length four. Moreover, it is easy to see that $a_2 \leq 2$ и $a_3 \leq 2$. Hence $a_2 = a_3 = 2$, $a_4 = 1$. Then $01110, 10001 \in B_1$ and it follows that B_1 does not contain a phone number with two blocks.

On the other hand, it is possible to find four phone numbers in B_1 which satisfy c). Take, for example, $B_1 = \{10001,\ 01010,\ 11100,\ 00111\}$.

The set C can be written as

$$C = A \cup B = A \cup \left(\cup_{0 \leq i < j \leq 9} B_{i,j} \cup B_{j,i}\right).$$

It is clear that for $(i, j) \neq (k, l)$ the choices of phone numbers in $B_{i,j} \cup B_{j,i}$ and $B_{k,l} \cup B_{l,k}$ are independent. Moreover, we may choose ten phone numbers in A which are not in conflict with any choice of the other numbers in C. Take, for example, $A = \{00000,\ 11111,\ \dots,\ 99999\}$.

Thus the maximal cardinality of C equals

$$|C| = |A| + \sum_{0 \leq i < j \leq 9} |B_{i,j} \cup B_{j,i}| = 10 + \binom{10}{2} \cdot 4 = 10 + 45 \cdot 4 = 190.$$

12.4. Set $AO = R$, $BD = b$, $CD = c$ and $OD = d$. Since CO is the bisector of $\measuredangle ACD$, then

$$\frac{d}{R} = \frac{c}{b + c}.$$

Let the line AO meet the circumcircle of $\triangle ABC$ at E. Then $AD.DO = BD.CD$, i.e.

$$(R + d)(R - d) = bc.$$

Since $d = \dfrac{cR}{b+c}$, it follows that $R^2 = \dfrac{(b+c)^2c}{b+2c}$. Set $k = (b, c, R)$, $m = \left(\dfrac{b}{k}, \dfrac{c}{k}\right)$, $R_1 = \dfrac{R}{k}$, $b_1 = \dfrac{b}{km}$ and $c_1 = \dfrac{c}{km}$. Then

$$R_1^2 = \frac{m^2(b_1+c_1)^2c_1}{b_1+2c_1}.$$

Since $(m, R_1) = 1$ and $(b_1 + 2c_1, b_1 + c_1) = (b_1 + 2c_1, c_1) = (b_1, c_1) = 1$, we get $R_1^2 = (b_1 + c_1)^2c_1$ and $m^2 = b_1 + 2c_1$. Hence c_1 is a perfect square, say $c_1 = n^2$. Now $c = kmc_1 = kmn^2$, $b = kmb_1 = km(m^2 - 2n^2)$ and $R = kR_1 = kn(m^2 - n^2)$.

The inequality $1 > \sin \measuredangle BAC = \dfrac{b+c}{2R} = \dfrac{m}{2n}$ shows that $\sqrt{2}n < m < 2n$. (Conversely, this condition implies that such a $\triangle ABC$ exists, it is acute and the line AO meets the side BC.) In particular, $n \geq 2$.

Since $R - c = kn(m^2 - n^2 - mn)$ is a prime number, it follows that n is a prime number, $k = 1$ and $m^2 - n^2 - mn = 1$, i.e. $(m-1)(m+1) = n(m+n)$. Hence n divides either $m - 1$ or $m + 1$.

1) Let $m - 1 = ln$. Then $l(ln + 2) = ln + 1 + n$, i.e.

$$n = \frac{1-2l}{l^2 - l - 1}.$$

Since $n < 0$ for $l \geq 2$ we get $l = 1$ and $n = 1$, a contradiction.

2) Let $m + 1 = ln$. Then $l(ln - 2) = ln - 1 + n$, i.e.

$$n = \frac{2l-1}{l^2 - l - 1}.$$

Since $n \leq 1$ for $l \geq 3$ and $n = -1$ for $l = 1$ we get $l = 2$. Then $n = R - c = 3$, $m = 5$, $b = 35$ and $c = 45$.

8.1. Let $ac - 3bd = 5$ and $ad + bc = 6$ for some integers a, b, c, d. Then

$$(a^2 + 3b^2)(c^2 + 3d^2) = (ac - 3bd)^2 + 3(ad + bc)^2 = 5^2 + 3.6^2 = 133 = 7.19.$$

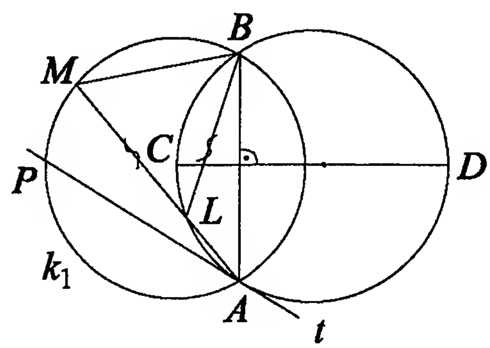
Because of the symmetry between the pairs (a, b) and (c, d) we consider only the cases when $a^2+3b^2 = 1$ and $c^2+3d^2 = 133$ or $a^2+3b^2 = 7$ and $c^2+3d^2 = 19$.

If $a^2 + 3b^2 = 1$, then $a^2 = 1$, $b = 0$ and using $ac = 5$, $ad = 6$ we get $c = 5$, $d = 6$ for $a = 1$ and $c = -5$, $d = -6$ for $a = -1$.

If $a^2 + 3b^2 = 7$, then $a^2 = 4$ and $b^2 = 1$. It follows by $c^2 + 3d^2 = 19$ that $c^2 = 16$ and $d^2 = 1$. Then $|ac| = 8$ and the equality $ac - 3bd = 5$ gives $ac = 8$. Therefore $a = 2$, $c = 4$ or $a = -2$, $c = -4$ and we get respectively $b = d = 1$ and $b = d = -1$.

Finally, all solutions are: $(a, b, c, d) = (\pm1, 0, \pm5, \pm6)$, $(\pm5, \pm6, \pm1, \pm0)$, $(\pm2, \pm1, \pm4, \pm1)$ and $(\pm4, \pm1, \pm2, \pm1)$.

8.2. Let CD be the diameter of k such that $CD \perp AB$ and let $L \in \overset{\frown}{ACB}$. Then $\sphericalangle ALB = 2\alpha$ is constant and we have $\sphericalangle AMB = \alpha$ since $\triangle MLB$ is isosceles. Therefore M belongs to an arc of the circle k_1, from which the segment AB is seen by the angle α.



Analogously, for $L \in \overset{\frown}{ADB}$ we have $\sphericalangle ALB = 180° - 2\alpha$, $\sphericalangle AMB = 90° - \alpha$ and we conclude that M belongs to an arc of the circle k_2, from which the segment AB is seen by the angle $90° - \alpha$. Let t be the tangent line to k at the point A and $t \cap k_1 = \{A, P\}$, $t \cap k_2 = \{A, Q\}$. Let λ be the half-plane with respect to t, containing k. Since L is between A and M we have $M \in \lambda$. Therefore the required locus consists of the arcs of k_1 and k_2 belonging to λ.

8.3. Assume the contrary. Then there exists a positive integer multiple of u_m such that the sum of its digits is less than m and let t be the smallest number with this property. Since $t > 10^m$, the number t can be written as $t = 10^m a + b$, where $0 < b < 10^m$.

We have $t = 10^m a + b = (10^m - 1)a + a + b$. Since u_m divides both t and $10^m - 1 = 9u_m$, we conclude that u_m divides $a + b$. But the sum of the digits of $a + b$ does not exceed the sum of the digits of t and $a + b < t$, a contradiction.

8.4. Denote by $A_1, A_2, \dots, A_5$ and $B_1, B_2, \dots, B_5$ the countries on the respective sides of the sheet of paper. Let S_{ij} be the area of the part of A_i which belongs to the country B_j on the other side of the sheet. (If A_i and B_j do not have a common area, then $S_{ij} = 0$.) Then, setting the area of the sheet to be 1, we have

$$\begin{aligned} S &= (S_{11} + S_{12} + S_{13} + S_{14} + S_{15}) + (S_{21} + S_{22} + S_{23} + S_{24} + S_{25}) + \cdots \\ &\quad + (S_{51} + S_{52} + S_{53} + S_{54} + S_{55}) = 1, \end{aligned}$$

since $S_{i1}+S_{i2}+S_{i3}+S_{i4}+S_{i5}$ equals the area of A_i.

The sum S can be written also as follows:

$$\begin{aligned} S &= (S_{11}+S_{22}+S_{33}+S_{44}+S_{55})+(S_{12}+S_{23}+S_{34}+S_{45}+S_{51})+\cdots \\ &\quad +(S_{15}+S_{21}+S_{32}+S_{43}+S_{54}). \end{aligned}$$

Hence at least one of the summands is greater than or equal to 0, 2 and let us assume that $S_{13}+S_{24}+S_{35}+S_{41}+S_{52}\geq 0,2$. We now color the countries $B_1, B_2, \ldots, B_5$ as follows: B_3 by the color of A_1, B_4 by the color of A_2, B_5 by the color of A_3, B_1 by the color of A_4 and B_2 by the color of A_5. Then every two countries are colored in different colors and at least 20% of the sheet is colored in the same color on both sides.

9.1. Using Vieta's formulas we get

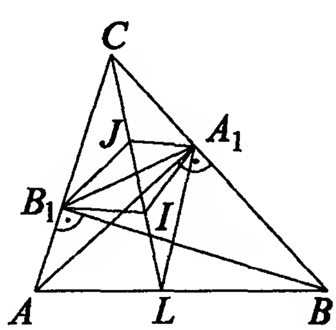
$$\frac{1}{x_1-2}+\frac{1}{x_2-2}=\frac{x_1+x_2-4}{(x_1-2)(x_2-2)}=-\frac{a+4}{3a^2-5a-15}.$$

Therefore $3a^2-5a-15\neq 0$ and

$$\frac{a+4}{3a^2-5a-15}=\frac{2a}{13}.$$

Hence $6a^3-10a^2-43a-52=0 \iff (a-4)(6a^2+14a+13)=0$, i.e. $a=4$. In this case $x_{1,2}=-2\pm\sqrt{15}$.

9.2. a) Since $\sphericalangle CB_1A_1 = \sphericalangle CIA_1$ and the quadrilateral ABA_1B_1 is cyclic, we have $\sphericalangle CIA_1 = \sphericalangle CB_1A_1 = \sphericalangle ABC$. This shows that the quadrilateral LBA_1I is cyclic. Then



$$\begin{aligned} \sphericalangle LA_1B &= \sphericalangle LIB = \sphericalangle ICB+\sphericalangle IBC \\ &= \frac{1}{2}(\sphericalangle ACB+\sphericalangle ABC) \\ &= \frac{1}{2}(\sphericalangle ACB+\sphericalangle A_1B_1C) \\ &= \frac{1}{2}\sphericalangle BA_1B_1. \end{aligned}$$

Hence A_1L is the bisector of $\sphericalangle BA_1B_1$, which completes the proof.

b) If J is the midpoint of CI, then we have $CJ=JI=IL$. But $IL=IA_1$ from a) and we conclude that JLA_1 is a right triangle. Then A_1J is the bisector of $\sphericalangle B_1A_1C$ and therefore J is the incenter of $\triangle A_1B_1C$. On the other hand, we have $\triangle A_1B_1C\sim\triangle ABC$, whence $\dfrac{A_1C}{AC}=\dfrac{CJ}{CI}=\dfrac{1}{2}$ and $\sphericalangle ACB=60°$.

Remark. One can prove that in case b) $\triangle ABC$ is equilateral.

9.3. The elements of A give $\binom{k}{2} + k + k = \dfrac{k(k+3)}{2}$ sums of the required kind (with two different summands, two equal summands or one summand, respectively). Therefore $\dfrac{k(k+3)}{2} \geq 27$, whence $k \geq 6$.

If $k = 6$ every element of M has a unique representation. This consecutively implies $1 \in A$, $2 \notin A$, $3 \in A$, $4 \notin A$, $5 \in A$ and now $6 = 3+3 = 1+5$ has two representations, a contradiction. Therefore $k \geq 7$.

Assume that $A = \{a_1, a_2, \ldots, a_7\}$ has the required property and $a_1 < a_2 < \cdots < a_7$. Since 1 and 27 have unique representations, we have $a_1 = 1$, $a_6 = 13$ and $a_7 = 14$. Also, it is easy to see that $a_2 \in \{2, 3\}$ and $a_3 \leq 5$. The only two possible representations of 25 are $25 = 13+12 = 14+11$ and therefore $a_5 = 12$ or $a_5 = 11$.

Case 1. $a_5 = 12$. It follows from $23 = 14+9 = 13+10 = 12+11$ that $a_4 \in \{9, 10, 11\}$. Then checking all possibilities for 21 we see that $a_3 \geq 7$, which contradicts to the above restriction $a_3 \leq 5$.

Case 2. $a_5 = 11$. As in case 1 we conclude that $a_4 \in \{9, 10\}$. On the other hand by $21 = 14+7 = 13+8 = 11+10$ we have $a_4 \in \{7, 8, 10\}$ and therefore $a_4 = 10$. Now using $a_3 \leq 5$ we see that the only possibility for 19 is $19 = 14+5$. This implies that $a_3 = 5$ and $a_2 = 3$, i.e. $A = \{1, 3, 5, 10, 11, 13, 14\}$. But this set does not have the required property.

The set $A = \{1, 3, 5, 7, 9, 11, 13, 14\}$ has cardinality 8 and possesses the required property. Therefore the minimum value of k is 8.

Remark. The consideration of the case $k = 6$ is not obligatory, it only gives an argument which leads to the solution.

9.4. If $f(n) = n-1$, then n divides the sum $1+2+\cdots+(n-1) = \dfrac{(n-1)n}{2}$, which implies that n is odd.

The numbers $n = p^s$, where $p > 2$ is a prime number and $s \geq 1$, are solutions. Indeed, if $k \in \mathbb{N}$ and $k < p^s - 1$, then the sum $1+2+\cdots+k = \dfrac{k(k+1)}{2}$ is not divisible by p^s because k and $k+1$ are coprime and less than p^s.

We shall prove that these are the only solutions. Let n be an odd positive integer which is not a power of a prime number. Then $n = ab$, where $a > 1$, $b > 1$ and $(a, b) = 1$. By the Chinese Remainder Theorem we conclude that there exists an integer $k \in [0, ab-1]$ such that $a|k$ and $b|k+1$. It is clear that $k \neq 0$, $k \neq ab-1$ and $ab|\dfrac{k(k+1)}{2}$. Therefore $n = ab$ is not a solution.

10.1. a) The equation (1) can be written as $9^x = 2^x$, i.e. $\left(\dfrac{9}{2}\right)^x = 1$ and $x = 0$.

b) We plug the only solution $x = 0$ of (1) in (2) and obtain $|a-1| = 1-a \iff a \leq 1$. In this case 0 is a solution of (2) and we have to decide when (2) has no other solution(s).

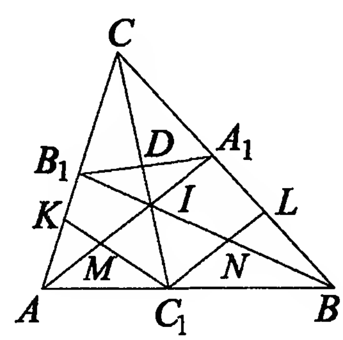
Set $5^x = t$, $t > 0$. Then we have to find all $a \leq 1$ such that the equation

$$at^2 + (1-a)t - 1 = 0 \tag{1}$$

has not positive roots different from 1. For $a = 0$ the only root of (3) is $t = 1$ and the condition is satisfied. For $a \neq 0$ the roots of (3) are 1 and $-\frac{1}{a}$. Therefore the condition is satisfied if and only if $-\frac{1}{a} = 1$ or $-\frac{1}{a} < 0$. Hence $a = -1$ or $a > 0$ and we conclude that $a \in [0, 1] \cup \{-1\}$.

10.2. a) We have $CA' = \frac{ab}{b+c}$ and $\frac{AI}{IA'} = \frac{AC}{CA'} = \frac{b+c}{a}$. By the Menelaus theorem for $\triangle AIC$ and the line $B'A'$ we have

$$\frac{CD}{DI} \cdot \frac{IA'}{A'A} \cdot \frac{AB'}{B'C} = 1 \Rightarrow \frac{CD}{DI} = \frac{a+b+c}{a} \cdot \frac{a}{c} = \frac{a+b+c}{c}.$$



b) By the Menelaus theorem for $\triangle AIC$ and the line KC' we get

$$\frac{AM}{MI} \cdot \frac{IC'}{C'C} \cdot \frac{CK}{KA} = 1,$$

whence $\frac{CK}{KA} = \frac{a+b+c}{c}$. Analogously, $\frac{CL}{LB} = \frac{a+b+c}{c}$.

Let h be the homothety with center C and ratio $\frac{a+b+c}{a+b+2c}$. Then $h(A) = K$, $h(B) = L$, $h(C) = C$ and it follows from a) that $h(I) = D$. Therefore D is the incenter of $\triangle KLC$.

10.3. If 39 thieves take 101 euro each and the last one takes 61 euro, then the only poor groups are those having as a member the last thieve. So this distribution of the money gives $\binom{39}{4}$ poor groups. We shall prove that this is the required minimum.

Let r be the number of all possible divisions of the thieves in 8 groups of five thieves each. Any such division has at least one poor group. The total number of the groups in all divisions is $8r$. Each group takes part in $8r/\binom{40}{5}$ divisions. Therefore each poor group is counted exactly $8r/\binom{40}{5}$ times and this gives at least r. Therefore we have at least $r\binom{40}{5}/8r = \binom{39}{4}$ poor groups.

10.4. See Problem 9.4.

11.1. Since $S_n = a_1 \cdot \dfrac{1-q^n}{1-q}$, the sequence $\{S_n\}_{n=1}^{\infty}$ converges if and only if $|q| < 1$. Therefore $-1 < \dfrac{3-2a}{a-2} < 1$, whence $a \in \left(1, \dfrac{5}{3}\right) \setminus \left\{\dfrac{3}{2}\right\}$. In this case

$$S = \lim_{n\to\infty} S_n = a_1 \cdot \frac{1}{1-q} = \frac{3-2a}{1-\dfrac{3-2a}{a-2}} = \frac{(3-2a)(a-2)}{3a-5}$$

and we have to prove that for every $a \in \left(1, \dfrac{5}{3}\right) \setminus \left\{\dfrac{3}{2}\right\}$ the inequality

$$\frac{(3-2a)(a-2)}{3a-5} < 1$$

holds. This inequality is equivalent to $\dfrac{2a^2-4a+1}{3a-5} > 0$ and since $3a-5<0$, we have to prove that $f(a) = 2a^2-4a+1 < 0$. This follows from $f(1) = -1$ and $f\left(\dfrac{5}{3}\right) = -\dfrac{1}{9}$.

11.2. The system is defined for $x \in (-\infty, -1] \cup [0, +\infty)$ and every y. We consider three cases.

Case 1. Let $\sin(\pi y) > 0$. The first equation gives

$$4^{\sqrt{x^2+x}} + 7.2^{\sqrt{x^2+x}} - 1 = 7$$

and setting $t = 2^{\sqrt{x^2+x}} > 0$ we obtain the equation $t^2 + 7t - 8 = 0$ with roots $t_1 = 1$ and $t_2 = -8$. Since $t > 0$, it follows that $2^{\sqrt{x^2+x}} = 1$, whence $x^2 + x = 0$, i.e. $x = 0$ and $x = -1$. For $x = 0$ we get $y = 0$ and for $x = -1$ we have $y = \pm\sqrt{3}$. Since $\sin 0 = 0$, $\sin(\sqrt{3}\pi) < 0$ and $\sin(-\sqrt{3}\pi) > 0$, the only solution of the system in this case is $x = -1$, $y = -\sqrt{3}$.

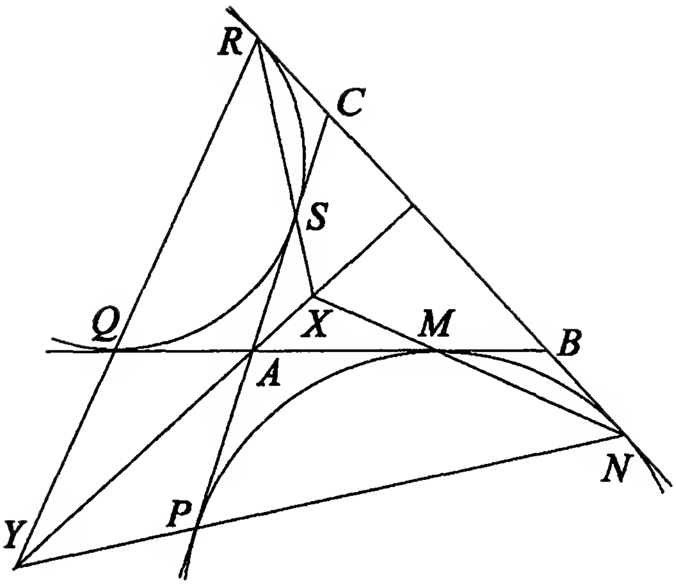
Case 2. Let $\sin(\pi y) = 0$, i.e. y is an integer. From the second equation we obtain $4 - y^2 \geq 0$, whence $y = 0, \pm1, \pm2$. For $y = 0$ we get $x = 0, -4$; for $y = \pm1$ we find $x = -2 \pm \sqrt{3}$ and for $y = \pm2$ we have $x = -2$. Since $-1 < -2 + \sqrt{3} < 0$, the solutions of the system in this case are $x = y = 0$, $x = -4$, $y = 0$, $x = -2 - \sqrt{3}$, $y = \pm1$ and $x = -2$, $y = \pm2$.

Case 3. Let $\sin(\pi y) < 0$. Then the first equation gives

$$4^{\sqrt{x^2+x}} + 7.2^{\sqrt{x^2+x}} - 1 = -7$$

and setting $t = 2^{\sqrt{x^2+x}} > 0$ we obtain the equation $t^2 + 7t + 6 = 0$ with roots $t_1 = -1$ and $t_2 = -6$. Since $t > 0$, the system has no solutions in this case.

11.3. We shall use the standard notations. We first prove that $YX \perp BC$. In $\triangle BQR$ we have $\sphericalangle BRY = 90^\circ - \dfrac{\beta}{2}$ and since $\sphericalangle MNB = \dfrac{\beta}{2}$ it follows that $NX \perp RY$. Analogously $RX \perp NY$. This means that X is the orthocenter of $\triangle NRY$ and $YX \perp RN$. Hence $YX \perp BC$.



Now we shall prove that X lies on the altitude of $\triangle ABC$ through A. Denote by X' its intersection point with MN. Since $BM = BN$ and $CP = CN = p$, we have $BM = BN = p - a$. Then $AM = c - (p - a) = p - b$ and by the Sine theorem for $\triangle AMX'$ we get

$$\frac{AX'}{\sin\frac{\beta}{2}} = \frac{AM}{\sin\sphericalangle AX'M} \iff \frac{AX'}{\sin\frac{\beta}{2}} = \frac{p-b}{\sin(90^\circ + \frac{\beta}{2})} \iff AX' = (p-b)\operatorname{tg}\frac{\beta}{2}.$$

On the other hand, if T is the tangent point of the incircle with the side BC, then $BT = p - b$ and $r = (p-b)\operatorname{tg}\frac{\beta}{2}$. Therefore $AX' = r$. Analogously, if X'' is the intersection point of RS and the altitude of $\triangle ABC$ through A, then $AX'' = r$. Therefore $X' \equiv X'' \equiv X$.

Since $YX \perp BC$ and $AX \perp BC$, we conclude that the points X, A and Y are colinear.

11.4. Let m and n be positive integers and $n = mq + r$, $0 \le r < m$. Then $\left[\frac{n+m-1}{m}\right] = q + 1 + \left[\frac{r-1}{m}\right]$ and $\frac{n}{m} = q + \frac{r}{m}$. Hence $\left[\frac{n+m-1}{m}\right] \ge \frac{n}{m}$ with equality if and only if m divides n.

Applying the above inequality we obtain

$$2.3^n = \prod_{i=1}^{k}\left[\frac{a_i + a_{i-1} - 1}{a_{i-1}}\right] \ge \prod_{i=1}^{k}\frac{a_i}{a_{i-1}} = \frac{a_k}{a_0} = 2.3^n.$$

Therefore a_{i-1} divides a_i for $i = 1, 2, \ldots, k$. We have to find the number of the sequences $1 = a_0 < a_1 < \cdots < a_k = 2.3^n$ such that a_{i-1} divides a_i for $i = 1, 2, \ldots, k$. Starting by an arbitrary sequence we construct a sequence of k symbols $\star$, n digits 3 and one digit 2 in the following way: if $\frac{a_{i+1}}{a_i} = 2^p 3^q$, $i = 1, 2, \ldots, k-1$ then we put p digits 2 and q digits 3 between $\star$ number i and $\star$ number $i+1$. Since $a_i < a_{i+1}$ there are no two consecutive symbols $\star$.

It is clear that any sequence of k symbols $\star$, n digits 3 and one digit 2 with no two consecutive symbols $\star$ corresponds to a sequence $1 = a_0 < a_1 < \cdots < a_k = 2.3^n$.

To count the number of the sequences of $\star$, 3 and 2 consider the sequence $\star\, 3\ 3\ 3 \ldots\ 3\ 3 \star$, where the number 3 is written n times. For a fixed ℓ, $0 \le \ell \le n-1$, we put ℓ stars between the 3's. This gives the distribution of the 3's in the sequence. The number 2 can be put on some $\star$ or between some consecutive stars – this give $2\ell + 3$ possibilities. Therefore the required number is equal to

$$\begin{aligned}\sum_{\ell=0}^{n-1}(2\ell+3)\binom{n-1}{\ell} &= 2\sum_{\ell=0}^{n-1}\ell\binom{n-1}{\ell} + 32\sum_{\ell=0}^{n-1}\ell\binom{n-1}{\ell} \\ &= 2(n-1)2\sum_{\ell=0}^{n-2}\binom{n-2}{\ell} + 3\sum_{\ell=0}^{n-1}\binom{n-1}{\ell} \\ &= 2(n-1)2^{n-2} + 3.2^{n-1} = (n+2)2^{n-1}.\end{aligned}$$

12.1. a) It follows by the recurrence relation that $x_1 = 2$, $x_2 = 1 + 2a$ and $x_3 = 1 + a + 2a^2$. Then $x_1 + x_3 = 2x_2 \iff 3 + a + 2a^2 = 2(1 + 2a)$ with solutions $a = 1$ and $a = \frac{1}{2}$. For $a = 1$ we get $x_{n+1} = x_n + 1$, i.e. the sequence is an arithmetic progression. For $a = \frac{1}{2}$ we see by induction on n that $x_n = 2$ for every n. Therefore $a = 1$ is the only solution.

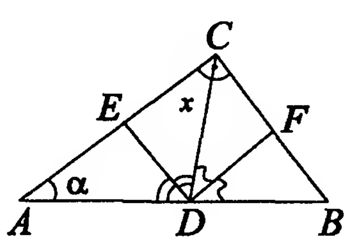
b) We prove by induction on n that $x_{n+1} = 1 + a + \cdots + a^{n-1} + 2a^n$, $n \ge 1$. For $a = 1$ we have $x_n = n + 1$, i.e. the sequence is not convergent. Let $a \ne 1$. Then

$$x_{n+1} = 2a^n + \frac{1-a^n}{1-a} = a^n\left(2 - \frac{1}{1-a}\right) + \frac{1}{1-a}.$$

If $2 - \frac{1}{1-a} = 0$, i.e. $a = \frac{1}{2}$, we get $x_{n+1} = 2$ for every n and the sequence is convergent.

Since $\{a^n\}_{n=1}^{\infty}$ converges if and only if $|a| < 1$ or $a = 1$, we conclude that the given sequence is convergent for $a \in (-1, 1)$ and its limit is equal to $\frac{1}{1-a}$ (since $\lim_{n\to\infty} a^n = 0$ for $a \in (-1, 1)$).

12.2. a) Set $\sphericalangle BAC = \alpha$ and $\sphericalangle ACD = x$. We have $AC = AB\cos\alpha$, $BC = AB\sin\alpha$ and it follows by the Sine theorem for $\triangle ADC$ and $\triangle BDC$ that



$$\begin{aligned}\frac{AC^2}{AD+CD}+\frac{BC^2}{BD+CD} &= \frac{AC}{\dfrac{AD}{AC}+\dfrac{CD}{AC}}+\frac{BC}{\dfrac{BD}{BC}+\dfrac{CD}{BC}}\\ &= \frac{AB\cos\alpha}{\dfrac{\sin x}{\sin(\alpha+x)}+\dfrac{\sin\alpha}{\sin(\alpha+x)}}+\frac{AB\sin\alpha}{\dfrac{\cos x}{\sin(\alpha+x)}+\dfrac{\cos\alpha}{\sin(\alpha+x)}}\\ &= AB\left(\frac{\cos\alpha\sin(\alpha+x)}{\sin\alpha+\sin x}+\frac{\sin\alpha\sin(\alpha+x)}{\cos\alpha+\cos x}\right)\\ &= AB\left(\frac{\cos\alpha\cos\dfrac{\alpha+x}{2}}{\cos\dfrac{\alpha-x}{2}}+\frac{\sin\alpha\sin\dfrac{\alpha+x}{2}}{\cos\dfrac{\alpha-x}{2}}\right)=AB.\end{aligned}$$

The above identity can be proved by using the Stewart theorem as well.
b) We have

$$\frac{AE}{CE}=\frac{AD}{CD}\Rightarrow\frac{AE+CE}{CE}=\frac{AD+CD}{CD}\Rightarrow CE=\frac{AC.CD}{AD+CD}$$

and similarly $CF=\dfrac{BC.CD}{BD+CD}$. Then

$$\begin{aligned}\frac{CF}{CA}+\frac{CE}{CB} &= \frac{BC.CD}{CA(BD+CD)}+\frac{AC.CD}{CB(AD+CD)}\\ &= \frac{CD}{CA.CB}\left(\frac{CA^2}{AD+CD}+\frac{BC^2}{BD+CD}\right)\end{aligned}$$

and using a) we see that

$$\frac{CF}{CA}+\frac{CE}{CB}=\frac{CD.AB}{CA.CB}.$$

Therefore the required minimum is equal to 1 and it is attained when CD is the altitude of $\triangle ABC$.

12.3. Let z_k, $1\le k\le 4$, be the roots of the given equation. Using the Vieta's formulas we obtain

$$z_1+z_2+z_3+z_4=a \quad\text{and}\quad z_1^2+z_2^2+z_3^2+z_4^2=a^2.$$

Set $u_k=\dfrac{2z_k}{a}=x_k+iy_k$, $1\le k\le 4$, where $x_k,y_k\in\mathbb{R}$. Then $u_k^2=x_k^2-y_k^2+2ix_ky_k$ and we get

$$x_1+x_2+x_3+x_4=2, \tag{1}$$
$$x_1^2+x_2^2+x_3^2+x_4^2=4+y_1^2+y_2^2+y_3^2+y_4^2\ge 4. \tag{2}$$

On the other hand,

$$|a-z_k|\ge|z_k| \iff |2-u_k|\ge|u_k| \iff (2-x_k)^2+y_k^2\ge x_k^2+y_k^2 \iff x_k\le 1.$$

Now (1) implies $x_k \geq -1$, i.e. $x_k^2 \leq 1$, which together with (2) gives $x_k = \pm 1$, $y_k = 0$.

Hence (1) shows that three of the numbers x_k are equal to 1 and the fourth one is -1. We can assume that $z_1 = z_2 = z_3 = -z_4$. Then $z_1 z_2 z_3 z_4 = -1$ implies $z_1 = \pm 1, \pm i$, which gives

$$(a, b) = (2, -2),\ (-2, 2),\ (2i, 2i),\ (-2i, -2i).$$

12.4. See Problem 11.4.

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9.1. It follows from Vieta's formulae that

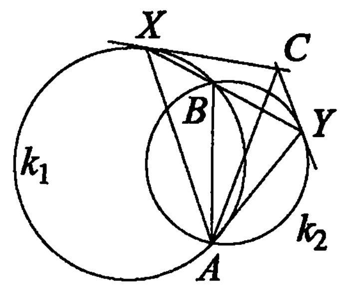
$$-6 = x_1 + x_2 = x_1 + x_1^3 - 8x_1.$$

Therefore $x_1^3 - 7x_1 + 6 = 0$ and $x_1 = -3, 1$ or 2. Plugging these values of x_1 in the initial equation gives $a = 3$ for $x = -3$, $a = -1$ and 7 for $x = 1$ and $a = -2$ and 8 for $x = 2$.

9.2. a) The quadrilateral $XCYA$ is cyclic since

$$\begin{aligned} \sphericalangle XCY &= 180^\circ - \sphericalangle CXY - \sphericalangle CYX = 180^\circ - \sphericalangle XAB - \sphericalangle BAY \\ &= 180^\circ - \sphericalangle XAY. \end{aligned}$$

Therefore $\sphericalangle XAC = \sphericalangle XYC = \sphericalangle BAY$.



b) It follows from a) that $\triangle XAC \sim \triangle BAY$. Therefore $\dfrac{XC}{AC} = \dfrac{BY}{AY}$ and since $XB = BY$ we have $\dfrac{XC}{XB} = \dfrac{AC}{AY}$. Moreover $\sphericalangle CXY = \sphericalangle CAY$, implying $\triangle XCB \sim \triangle ACY$. Thus, $\sphericalangle XBC = \sphericalangle AYC = \sphericalangle XBA$.

9.3. Setting $p = m - n$ and $q = m + n$ gives

$$mn = \frac{1}{4}(q^2 - p^2) \quad \text{and} \quad m^2 + n^2 = \frac{1}{2}(q^2 + p^2).$$

Hence the given conditions can be written as

$$8l^2 - 2q^2 + 2p^2 = q^2 + p^2 + 5ql,$$

i.e.

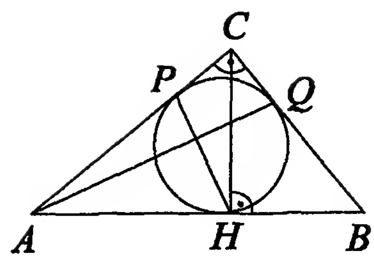
$$p^2 = (3q + 8l)(q - l).$$

Since p is a prime number and $3q + 8l > q - l$ we obtain $p^2 = 3q + 8l$ and $1 = q - l$. Hence $11l + 3 = p^2$.

9.4. Set $u = x - \dfrac{2}{x}$. Then the equation becomes

$$u^2 + 2u + a^2 - a - 5 = 0. \tag{1}$$

Since the equation $x^2 - ux - 2 = 0$ has real solutions for any real u, it suffices to find the integer values of a for which the equation (1) has a real root. The last holds when $D = -a^2 + a + 6 \geq 0$, i.e. $(a-3)(a+2) \leq 0$ giving $a \in [-2, 3]$. Therefore $a = -2, -1, 0, 1, 2, 3$.



9.5. It follows from $AQ \perp HP$ that $\sphericalangle QAB = \sphericalangle PHC$. On the other hand $\sphericalangle ABC = \sphericalangle ACH$ and therefore $\triangle ABQ \sim \triangle HCP$. Thus, $\dfrac{AB}{BQ} = \dfrac{HC}{CP}$. Using the standard notation for the elements of a triangle we obtain the following equalities:

$$\begin{aligned} \frac{c}{p-b} = \frac{h}{r} \;&\Leftrightarrow\; \frac{c}{p-b} = \frac{2S/c}{S/p} \;\Leftrightarrow\; \frac{c}{p-b} = \frac{2p}{c} \\ &\Leftrightarrow\; \frac{2c}{a+c-b} = \frac{a+b+c}{c} \;\Leftrightarrow\; 2c^2 = (a+c)^2 - b^2 \\ &\Leftrightarrow\; c^2 = a^2 + 2ac - b^2 \;\Leftrightarrow\; b^2 = ac, \end{aligned}$$

since $c^2 = a^2 + b^2$. Hence

$$b^4 = a^2(a^2+b^2) \iff b^4 - a^2b^2 - a^4 = 0.$$

Set $k = \dfrac{AH}{BH} = \dfrac{b^2/c}{a^2/c} = \dfrac{b^2}{a^2}$. Then $k^2 - k - 1 = 0$ and we get $k = \dfrac{1+\sqrt{5}}{2}$.

9.6. Consider a graph G whose vertices are the airports in the country. Two vertices form an edge if there is an airline between the corresponding airports. Suppose that a round trip satisfying the conditions of the problem does not exist, i.e. there is no a cycle of length 4 in G.

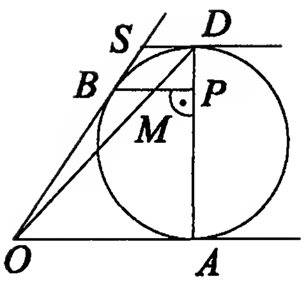
If x is a vertex of G denote by $d(x)$ the number of neighbors of x. Then the number of pairs both elements of which are neighbors of x equals $\dbinom{d(x)}{2}$. Note that every pair is counted from at most one vertex z, since otherwise there is a cycle of length 4.

Using the identity $\sum_{x\in G} d(x) = 72$ and the Root mean square – Arithmetic mean inequality we have

$$\begin{aligned} \binom{16}{2} = 120 \;&\geq\; \sum_{x\in G}\binom{d(x)}{2} = \sum_{x\in G}\frac{d^2(x)}{2} - \sum_{x\in G}\frac{d(x)}{2} \\ &\geq\; \frac{1}{32}\left(\sum_{x\in G} d(x)\right)^2 - \sum_{x\in G}\frac{d(x)}{2} \\ &=\; \frac{72^2}{32} - 36 = 126, \end{aligned}$$

a contradiction.

10.1. Let the tangent line to k at D meet the ray $\overrightarrow{OB}$ at point S. Then the lines SD, BP and OA are parallel and therefore $\triangle OBM \sim \triangle OSD$ and $\triangle DPM \sim \triangle DAO$. It follows that $\frac{BM}{SD} = \frac{OB}{OS}$ and $\frac{MP}{OA} = \frac{DP}{DA}$, i.e. $BM = \frac{OB.SD}{OS}$ and $MP = \frac{OA.DP}{DA}$.



Since $OA = OB$ and $\frac{BS}{OS} = \frac{DP}{DA}$ (we use the equality $SD = SB$), we obtain $BM = MP$, implying that the desired ratio equals $1 : 2$.

10.2. The domain of $f(x)$ is $x > 0$. Setting $y = \lg x$ gives

$$F(y) = \frac{2y^2 + 3y + 3}{y^2 + 2y + 2}.$$

Since the denominator is positive the function $F(y)$ is defined for all real y.

Let M be the desired value of $f(x)$ (if it exists). Then for any real y we have

$$\frac{2y^2 + 3y + 3}{y^2 + 2y + 2} \leq M,$$

$$2y^2 + 3y + 3 \leq My^2 + 2My + 2M,$$

$$(2 - M)y^2 + (3 - 2M)y + (3 - 2M) \leq 0.$$

Therefore $2-M < 0$ and $D = (3-2M)(3-2M-8+4M) = (3-2M)(2M-5) \leq 0$. Hence $M \geq 2.5$.

Note that for $M = 2.5$ the above inequality becomes $-0,5y^2 - 2y - 2 \leq 0$, i.e. $y^2 + 4y + 4 = (y + 2)^2 \geq 0$ and the equality is attained only if $y = -2$, i.e. for $x = 0,01$.

Therefore the maximum of the function equals $2,5$ and it is attained for $x = 0,01$ only.

10.3. Let $x = \frac{p}{q}$, where p and q are coprime positive integers. We shall prove by induction on $n = p + q \geq 2$ that $f(x)$ is uniquely determined. This is true for $n = 2$ (since $f(1) = 1$). Suppose that it is true for all integers less than a given $n \geq 3$ and consider $x = \frac{p}{q}$, where $p + q = n$. It follows from the first condition that we may assume that $p > q$.

Now the second condition shows that $f\left(\frac{p}{q}\right)$ is uniquely determined by $f\left(\frac{p-q}{q}\right)$ and since $p - q + q < n$ it is uniquely determined by the induction hypothesis.

Note that the function $f\left(\frac{p}{q}\right) = \frac{p+q}{2}$ for p and q relatively prime fulfills the conditions of the problem.

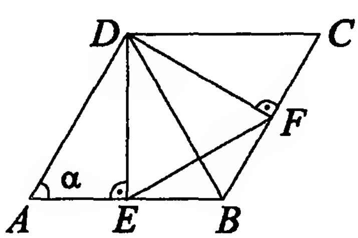
10.4. It follows from the condition of the problem that

$$\frac{100-x}{100} \cdot \frac{100+y}{100} = \frac{100-y+x}{100}.$$

Hence $(100-x)(100+y) = 100(100-y+x)$ which can be written as $(200-x)(200+y) = 200^2$. Further, $100 < 200-x < 200$, i. e. $200-x$ equals 125 or 160.

Therefore the solutions are $x_1 = 75$, $y_1 = 120$ and $x_2 = 40$, $y_2 = 50$, both of which fulfill the condition.

10.5. Set $\sphericalangle BAD = \alpha$, $AB = CD = a$, $AD = BC = b$ and $BD = d$. We have $DE = b\sin\alpha$, $AE = b\cos\alpha$, $DF = a\sin\alpha$ and $CF = a\cos\alpha$. Therefore $\Delta DEF \sim \Delta ADB$ and thus $EF = d\sin\alpha$.



Plugging the above expressions in the given inequality we see that it is equivalent to

$$4\sin\alpha + 4 - 4\sin^2\alpha \le 5,$$

i.e. $(2\sin\alpha - 1)^2 \ge 0$, which is true for any value of α. The equality holds when $\alpha = 30°$.

10.6. See problem 9.6.

11.1. Set $AB = c$, $BC = a$, $AC = b$, $p = \frac{a+b+c}{2}$ and let O, P and Q be the midpoints of AB, AM and BN, respectively.

Since $OO_1 \perp AC$ and $O_1P \perp AM$, we have $\Delta O_1OP \sim \Delta ABC$, implying that $\frac{OO_1}{c} = \frac{OP}{a}$. Further, it follows from $AM = p - c$ that

$$OP = OA + \frac{1}{2}AM = \frac{1}{2}c + \frac{1}{2}(p-c) = \frac{a+b+c}{4}$$

and therefore $OO_1 = \frac{c(a+b+c)}{4a}$. Analogously $OO_2 = \frac{c(a+b+c)}{4b}$.

Since $\sphericalangle ACB = 90°$ we have that $\sphericalangle O_1OO_2 = 90°$.

Finally, using that $S_{O_1O_2C} = |S_{OO_1O_2} - S_{OO_1C} - S_{OO_2C}|$ we compute

$$\begin{aligned} S_{OO_1O_2} - S_{OO_1C} - S_{OO_2C} &= \frac{OO_1.OO_2}{2} - \frac{OO_1.b}{4} - \frac{OO_2.a}{4} \\ &= \frac{c^2(a+b+c)^2}{32ab} - \frac{c(a+b+c)}{16}\left(\frac{b}{a} + \frac{a}{b}\right) \\ &= \frac{c^2(a+b+c)^2}{32ab} - \frac{c^3(a+b+c)}{16ab} = \frac{c^2}{16}, \end{aligned}$$

since $c^2 = a^2 + b^2$. This implies that $S_{O_1CO_2}$ does not depend on the choice of the point C.

11.2. For $t = 0$ the inequality is true. When $t \neq 0$ we set $u = \dfrac{1}{t}$ and we have to prove that

$$f(u) = 3u^2 + 2u(x+y+z) + xy + yz + zx \geq 0$$

for $|u| \geq 1$.

The latter inequality follows from the fact that the abscissa $-\dfrac{x+y+z}{3}$ of the vertex of the the parabola $w = f(u)$ lies in the interval $[-1, 1]$ and that

$$f(\pm 1) = (x \pm 1)(y \pm 1) + (y \pm 1)(z \pm 1) + (z \pm 1)(x \pm 1) \geq 0$$

for $x, y, z \in [-1, 1]$. The equality occurs only if $f(\pm 1) = 0$, i.e. when $u = t = \pm 1$ and two of the numbers x, y and z are equal to ∓ 1, and the third one is arbitrary.

11.3. Consider a graph with vertices the given points. Two points form an edge if the corresponding pair of points is "isolated". We first prove that the graph is connected. To do this suppose that it has more than one connected component and chose points A and B from different components such that the distance AB is the least possible. Then the disk with diameter AB does not contain other vertices and therefore A and B are connected by an edge. This contradicts the choice of A and B.

Since a connected graph with 2006 vertexes has at least 2005 edges (if the graph is a tree) we conclude that there are at least 2005 "isolated" pairs.

If we take 2006 points on a semicircle such that the distances between consecutive points are equal then "isolated" are only the pairs of neighboring points. Therefore there are 2005 "isolated" pairs.

11.4. When $a = 1, 2, 3$ the system has solutions $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$, respectively. We shall prove that when $a = 4$ the system has no integer solution.

Suppose the contrary. Then we have

$$(1) \qquad 4 - z^2 = x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (4 - z)(x^2 - xy + y^2),$$

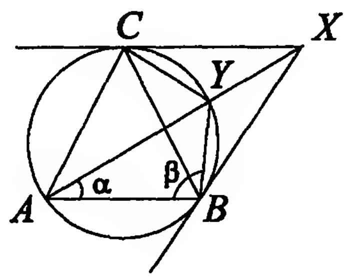
giving (since $z = 4$ does not lead to an integral solution) that $\dfrac{4 - z^2}{4 - z}$ is an integer. Since $\dfrac{4 - z^2}{4 - z} = \dfrac{-12 + 16 - z^2}{4 - z} = 4 + z + \dfrac{12}{z - 4}$, we conclude that $z - 4$ is a divisor of 12. Hence $z - 4 = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ and therefore $z = -8, -2, 0, 1, 2, 3, 5, 6, 7, 8, 10$ or 16. Using (1) we have

$$(x+y)^2 - 3xy = \frac{4 - z^2}{4 - z} \iff 3xy = (4 - z)^2 - \frac{4 - z^2}{4 - z}$$

and we obtain the following system for x and y:

$$\left| \begin{aligned} & x + y = 4 - z \\ & xy = \frac{(4-z)^3 + z^2 - 4}{3(4-z)} \end{aligned} \right. .$$

It is easy to check that all the values of z listed above do not lead to an integral solution for x and y.



11.5. If $\measuredangle BAY = \alpha$ and $\measuredangle ABY = \beta$ then $\measuredangle BYX = \beta + \alpha$. Furthermore $\measuredangle ACB = \measuredangle AYB = 180° - \alpha - \beta$, implying $\measuredangle BAC = \measuredangle ABC = \dfrac{\beta+\alpha}{2}$. Thus $\measuredangle AYC = \dfrac{\beta+\alpha}{2}$ and $\measuredangle YCX = \measuredangle YAC = \dfrac{\beta+\alpha}{2} - \alpha = \dfrac{\beta-\alpha}{2}$. The Sine theorem for $\triangle BYX$ and $\triangle CYX$ gives

$$\frac{XY}{XB} = \frac{\sin\alpha}{\sin(\alpha+\beta)}, \quad \frac{XY}{XC} = \frac{\sin\dfrac{\beta-\alpha}{2}}{\sin\dfrac{\beta+\alpha}{2}}$$

and since $XB = XC$ we have $\dfrac{\sin\alpha}{\sin(\alpha+\beta)} = \dfrac{\sin\dfrac{\beta-\alpha}{2}}{\sin\dfrac{\beta+\alpha}{2}}$. Hence

$$\sin\alpha = 2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2} = \sin\beta - \sin\alpha$$

and therefore $\dfrac{AY}{BY} = \dfrac{\sin\beta}{\sin\alpha} = 2$.

11.6. a) It follows from $a_n < 1$ and $a_{n+1} = \dfrac{3}{a_n+2} < 1$ that $a_n < -2$. Thus, $a_{n+1} < -2$, and therefore $a_n + 2 = \dfrac{3}{a_{n+1}} > -\dfrac{3}{2}$, i.e. $a_n > -\dfrac{7}{2}$.

б) *First solution.* Set $b_n = a_n + 3$. Then $b_{n+1} = \dfrac{3b_n}{b_n - 1}$. It follows from a) that $-\dfrac{3}{2} < b_n - 1 < 0$, i.e. $|b_n - 1| < \dfrac{3}{2}$ and therefore $|b_{n+1}| > 2|b_n|$. Hence $1 > |b_{n+1}| > 2^n|b_1|$ for all n. Letting $n \to \infty$ gives $b_1 = 0$. Then $b_n = 0$, i.e. $a_n = -3$ for all n.

Second solution. Let $c_1 = 1$ and $c_{k+1} = 3c_k + (-1)^k$ for $k \geq 1$. One can prove by induction on k that

$$(*) \qquad -\frac{c_{2k+1}}{c_{2k}} < a_n < -\frac{c_{2k}}{c_{2k-1}}$$

for all n and k. Indeed, similar to a) it follows from $a_n < -\frac{c_{2k}}{c_{2k-1}}$ that $-\frac{c_{2k+1}}{c_{2k}} < a_n$, which in turn implies that $a_n < -\frac{c_{2k+2}}{c_{2k+1}}$.

Further, induction on k gives $c_k = \frac{3^k + (-1)^{k-1}}{4}$. Therefore $\lim_{k\to\infty} \frac{c_{k+1}}{c_k} = 3$ and using $(*)$ we conclude that $a_n = -3$ for all n.

12.1. The common points of the graphs of the line and the parabola are $A(1,2)$ and $B(4,5)$. The equations of the tangents to the graph of the parabola at A and B are $y = -2x + 11$ and $y = 4x - 11$, respectively. The intersecting point of the two tangents is the point $C\left(\frac{5}{2}, -1\right)$. The area of $\triangle ABC$ then equals $\frac{27}{4}$.

12.2. See Problem 11.5.

12.3. *First Solution.* Write the equation in the form

$$x^2(x+a)^2 > 4x - 3. \tag{1}$$

Then for $x = 1$ we get $(a+1)^2 > 1$, i.e. $a > 0$ or $a < -2$. If $a < -2$, then $x = -a$ gives a contradiction $0 > -4a - 3$. Thus, $a > 0$.

Conversely, if $a > 0$, then (1) is satisfied for all x. Indeed, when $x \le 0$ this is obvious and when $x > 0$ we have $x^2(x+a)^2 > x^4 \ge 4x + 3$, since the later inequality is equivalent to $(x-1)^2((x+1)^2 + 2) \ge 0$.

Second Solution. It follows from (1) that we have to find all a, for which

$$a < -\frac{\sqrt{4x-3}}{x} - x = f(x)$$

for all $x \ge \frac{4}{3}$ or

$$a > \frac{\sqrt{4x-3}}{x} - x = g(x)$$

for all $x \ge \frac{4}{3}$.

The first case is impossible since $\lim_{x\to+\infty} f(x) = -\infty$. The maximum of the function g equals 0 (and it is attained for $x = 1$). Therefore the answer is $a > 0$.

12.4. The equality is equivalent to

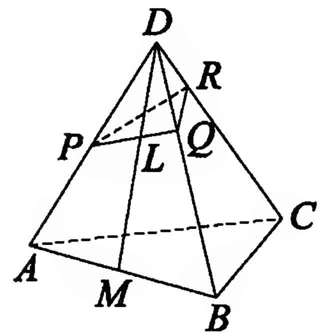
$$\sin(n-1)\alpha = \frac{(n-1)\sin 2\alpha}{2}. \tag{1}$$

When $n \ge 4$ setting $\alpha = \frac{\pi}{4}$ gives

$$\sin\left((n-1)\frac{\pi}{4}\right) = \frac{n-1}{2} \ge \frac{3}{2},$$

a contradiction.

When $n = 1$ and $n = 3$ the equality (1) is an identity and when $n = 2$ we have $\sin\alpha = \dfrac{\sin(2\alpha)}{2}$, which is not true for $\alpha = \dfrac{\pi}{4}$. Therefore the answer is $n = 1$ and $n = 3$.



12.5. Let the plane meet the edges DA, DB and DC at points P, Q and R, respectively. Set

$$\frac{DP}{DA} = x, \quad \frac{DQ}{DB} = y \quad \text{and} \quad \frac{DR}{DC} = z.$$

Let M be the midpoint of AB and $L = DM \cap PQ$. It follows from the condition of the problem that $\dfrac{DL}{DM} = \dfrac{1}{3}$. Therefore

$$\frac{S_{DLP}}{S_{DAM}} = \frac{DP.DL}{DA.DM} = \frac{x}{3}; \quad \frac{S_{DLQ}}{S_{DMB}} = \frac{DL.DQ}{DM.DB} = \frac{y}{3}.$$

Since $S_{DAM} = S_{DMB} = \dfrac{1}{2}S_{DAB}$ we conclude that

$$2xy = 2\frac{DP.DQ}{DA.DB} = \frac{S_{DPQ}}{\frac{1}{2}S_{DAB}} = \frac{S_{DPL}}{S_{DAM}} + \frac{S_{DLQ}}{S_{DMB}} = \frac{x+y}{3},$$

i.e. $\dfrac{1}{x} + \dfrac{1}{y} = 6$. Analogously $\dfrac{1}{y} + \dfrac{1}{z} = 8$ and $\dfrac{1}{z} + \dfrac{1}{x} = 10$. Solving this system we obtain $x = \dfrac{1}{4}$, $y = \dfrac{1}{2}$ and $z = \dfrac{1}{6}$. Thus

$$\frac{V_{DPQR}}{V_{DABC}} = \frac{DP.DQ.DR}{DA.DB.DC} = xyz = \frac{1}{48},$$

and therefore the desired ratio equals $1 : 47$.

12.6. See Problem 11.6.

55. Bulgarian Mathematical Olympiad
National round

1. Let B be a subset of A having the given property. Since $1+3=2^2$, we have that exactly one of the numbers 1 or 3 belongs to B.

If $1 \in B$ then $3 \notin B$. We prove by induction that for any integer t, $0 \leq t < 2^{n-2}$, the integers of the form $4t+1$ belong to B and the integers of the form $4t+3$ do not belong to B. The statement is true for $t=0$ and suppose it is true for $t \leq s$. Since $4(s+1)+1$ is an odd number there exists l such that $2^l < 4(s+1)+1 < 2^{l+1}$. Therefore $2(4s+5) > 2.2^l = 2^{l+1}$ giving $0 < 2^{l+1}-(4s+5) < 4s+5$. Set $x = 4s+5$ and $y = 2^{l+1}-(4s+5)$. Then $x+y = 2^{l+1}$ and since y is of the form $4m+3$ we conclude that $y \notin B$ and therefore $4(s+1)+1 \in B$. Analogously $4(s+1)+3 \notin B$.

If $1 \notin B$ then $3 \in B$ and we prove as above that the integers of the form $4t+1$ belong to B and the integers of the form $4t+3$ do not.

Therefore the odd numbers in B are either all integers of the form $4t+1$ or all integers of the form $4t+3$.

Let $x = 2^p x_0$ and $y = 2^q y_0$, where x_0 and y_0 are odd and p and q are positive integers. If $2^p x_0 + 2^q y_0 = 2^k$ and $p \neq q$, say $p < q$, then $x_0 + 2^{q-p} y_0 = 2^{k-p}$, which is impossible. Therefore $p = q$ and it follows that the sum of the elements from distinct sets $A_i = \{2^i a : a \text{ is an odd integer}\}$, $i = 1, 2, \dots n$ is not a power of 2. For any A_i, after dividing by 2^i and applying the above arguments, we obtain that either all integers of the form $2^i(4t+1)$ are in B or all integers of the form $2^i(4t+3)$ are in B.

Therefore there exist 2^{n+1} sets B with the given property.

2. a) It follows from $f(x+y)-f(x-y) > 0$ that f is an increasing function. Therefore the function $f(x)$ has a limit $l \geq 0$ when $x \to 0$, $x > 0$ (prove!). Thus letting $x, y \to 0$, $x > y > 0$, we get $l - l = 4\sqrt{l^2}$, i.e. $l = 0$. Fixing x and letting $y \to 0$, $y > 0$ we conclude that $f(x+y) - f(x-y) \to 0$. Since the function f is increasing we conclude that it is continuous at x. Finally, letting $y \to x$, $y < x$, we get $f(2x) = 4f(x)$.

b) Setting $x = ny > 0$, where $n \geq 2$ is an integer, we obtain from the given identity that

$$f((n+1)y) = f((n-1)y) + 4\sqrt{f(ny)f(y)}.$$

Using $f(2y) = 4f(y)$, it follows by induction that $f(ny) = n^2 f(y)$. Set $f(1) = c > 0$. Then $f(n) = n^2 c$. Now, for any positive integers p and q we have $cp^2 = f(q.p/q) = q^2 f(p/q)$, i.e. $f(p/q) = c(p/q)^2$. Since f is a continuous function, we conclude that $f(x) = cx^2$ for any $x > 0$. Conversely, any function of the form $f(x) = cx^2$ satisfies the condition.

Remark. It is possible to show that any function satisfying the condition is differentiable. Thus, letting $y \to 0$, $y > 0$, in the identity

$$\frac{f(x+y) - f(x-y)}{2y} = 2\frac{\sqrt{f(y)}}{y}\sqrt{f(x)}$$

we get $f'(x) = 2c\sqrt{f(x)}$. Therefore $(\sqrt{f(x)})' = c$, i.e. $f(x) = c^2x^2$.

3. Denote by M_n the set of all digits of the numbers $1, 2, \ldots, n$. First we find the least positive integer $n = \overline{a_1a_2\ldots a_t}$ such that every two nonzero digits appear different number of times in M_n. By adding zeros on the left we may assume that all numbers $1, 2, \ldots, n-1$ are t-digit numbers.

It is clear that every nonzero digit appears the same number of times. Let B_i^j, $1 \le i \le t$, $1 \le j \le 9$ be the number of appearances of the digit j in position i among the numbers $1, 2, \ldots, n$. Note that for all i and $j \le 8$, if a number A has $j+1$ in position i, then replacing this digit by j we obtain a number which is less than A. Therefore $B_i^j \ge B_i^{j+1}$.

Furthermore for a fixed i the inequality $B_i^j \ge B_i^{j+1}$ is fulfilled for at most two pairs of digits j and $j+1$, namely a_{i-1} and a_i; a_i and a_{i+1}. Moreover, if $i = t$, it is fulfilled only for a_t and a_{t+1}. Since there are 8 pairs of the form $(j, j+1)$ we have $t \ge 5$. If $n = 13578$ then $B_1^1 > B_1^2$; $B_2^2 > B_2^3$; $B_2^3 > B_2^4$; $B_3^4 > B_3^5$; $B_3^5 > B_3^6$; $B_4^6 > B_4^7$; $B_4^7 > B_4^8$; $B_5^8 > B_5^9$, i.e. $n = 13578$ satisfies the condition of the problem.

If $m < 13578$ also satisfies the condition then the first digit of m is 1 and the second digit is $0, 1, 2$ or 3. Since $B_1^j > B_1^{j+1}$ is true only for $j = 1$ if the second digit is 0, 1 or 2 then at least two consecutive digits appear equal number of times. Therefore the second digit of m is 3. It follows by similar arguments that the third, fourth and the fifth digits of m are respectively 5, 7 and 8. Therefore $n = 13578$ is the least positive integer such that every two nonzero digits appear different number of times in M_n.

Since the number of digits of all numbers $1, 2, 3, \ldots, 13578$ equals

$$9.1 + 90.2 + 900.3 + 9000.4 + 3579.5 = 56784$$

we conclude that 56784 has the desired property.

Suppose that there exists $k < 56784$ which has the desired property. Then the digits in the sequence are those in M_s for some $s < 13578$ and some digits of $s+1$. According to the previous observations there exist two consecutive digits that are not digits of s (eventually excluding the last one) appearing equal number of times in M_s. If the last digit of s is not 9, then the same digits appear equal number of times in the sequence since the digits of s and $s+1$ are the same (except the last one). If the last digit of s is 9 then the last digit of $s+1$ is 0 and therefore $s+1 < 13578$. Hence we conclude as above that there exist two consecutive digits not among the digits of $s+1$ which appear equal number of times.

4. Since $p-1$ is a divisor of $p!$ the greatest common divisor of $p-1$ and $p! + 2^n$ is a power of two. We shall show that both numbers $p-1$ and $p! + 2^n$ have at least one odd divisor.

Suppose that $p - 1 = 2^k$, i.e. $p = 2^k + 1$. If $s \ge 3$ is an odd divisor of k then

$p = 2^{st} + 1 = (2^t + 1)A$, i.e. p is not a prime number. Therefore $k = 2^t$ giving

$$\begin{aligned} 2^{p-1} - 1 &= 2^{2^k} - 1 = (2^{2^{k-1}} - 1)(2^{2^{k-1}} + 1) = \cdots \\ &= (2^{2^t} - 1)(2^{2^t} + 1)(2^{2^{t+1}} + 1) \ldots (2^{2^{k-1}} + 1). \end{aligned}$$

It is clear that p^2 does not divide the above product since $(2^{2^t} + 1, 2^{2^l} + 1) = 1$ when $l > t$, and $2^{2^t} - 1 < p = 2^{2^t} + 1$. Therefore $p - 1$ is not a power of 2.

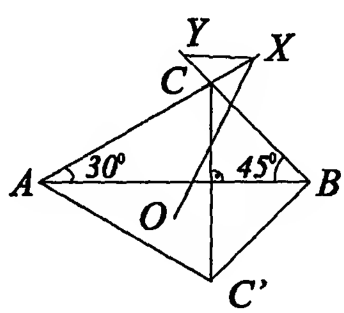
Suppose that $p! + 2^n = 2^k$, giving $k > n$ and $p! = 2^n(2^{k-n} - 1)$. Then p is a divisor of $2^m - 1$, where $m = k - n$. Let t be the least positive integer such that p divides $2^t - 1$. Then t is a divisor of m and t is a divisor of $p - 1$. If $p - 1 = lt$ then

$$2^{p-1} - 1 = (2^t - 1)(2^{t(l-1)} + 2^{t(l-2)} + \cdots + 2^t + 1).$$

Since $2^t \equiv 1 \pmod p$ we have $2^{t(l-1)} + 2^{t(l-2)} + \cdots + 2^t + 1 \equiv l \not\equiv 0 \pmod p$. Therefore p^2 is a divisor of $2^t - 1$ which implies that p^2 is a divisor of $2^m - 1$, i.e. p^2 is a divisor of $p!$, a contradiction.

Thus, both $p - 1$ and $p! + 2^n$ have at least one odd divisor and these divisors are distinct. Therefore the product $(p - 1)(p! + 2^n)$ has at least three distinct prime divisors.

5. We shall prove that the perpendicular bisector of XY passes through the point C' which is symmetric to the point C with respect of AB. Denote by R the circumradius of $\triangle ABC$. The Sine theorem for $\triangle ABC$ gives $AC = R\sqrt{2}$ and $BC = R$. Suppose that C lies between A and X. Then $OX = BY$, $C'A = CA$, $C'B = CB$, $\sphericalangle C'BY = 90°$ and the Cosine theorem for $\triangle C'AX$ gives



$$\begin{aligned} C'X^2 &= C'A^2 + AX^2 - C'A.AX = AC^2 + AX^2 - AC.AX \\ &= AC^2 + XA.XC = 2R^2 + (OX^2 - R^2) = R^2 + BY^2 \\ &= C'B^2 + BY^2 = C'Y^2. \end{aligned}$$

Therefore C' is a point on the perpendicular bisector of XY. In the case when X lies between A and C the proof is similar.

6. We first prove the following

LEMMA. *If $A \in S$, then the open disc $k(O, OA)$ is contained in S.*

Proof of the Lemma. Note that if $A \in S$ and B is a point on the circle of diameter OA then $B \in S$ (since B belongs to the circle with diameter OX, where $OB \perp BX$ and X is a point on the circle with diameter OA). Let $B \in k(O, OA)$ and $\varphi = \sphericalangle AOB$. For any positive integer n set $A_0 = A$ and define A_k, $k = 1, \ldots, n$, such that $\sphericalangle A_{k-1}OA_k = \dfrac{\varphi}{n}$ and $OA_k = OA_{k-1} \cos \dfrac{\varphi}{n}$.

Since $\sphericalangle OA_kA_{k-1} = 90°$, it follows by induction on k that $A_k \in S$, $k = 1, \ldots, n$; in particular $A_n \in k(O, OA)$. Since $B \in OA^{\rightarrow}$ and $OB < OA$, the statement of the Lemma would follow if $\lim\limits_{n\to\infty} OA_n = OA$. But the latter is true because $OA_n = OA\left(\cos\dfrac{\varphi}{n}\right)^n$ and $1 \geq \left(\cos\frac{\varphi}{n}\right)^{2n} = \left(1 - \sin^2\frac{\varphi}{n}\right)^n \geq \left(1 - \frac{\varphi^2}{n^2}\right)^n \to 1$.

It follows from the Lemma that if the set S is unbounded then $S = \mathbb{R}^2$. If S is bounded then setting $r = \sup\limits_{A\in S} OA$ we get $k(O,r) \subset S \subset \overline{k(O,r)}$. Therefore the desired sets are $S = \mathbb{R}^2$ and $S = k \cup \gamma$, where k is an open disc with center O, and γ is an arbitrary set on its circumference.

Team selection tests for 23. BMO

1. We shall prove that if the angles of two triangles form an arithmetic progression with difference d, then $d = 0$.

Let α be the least of all six angles and let the progression be $\alpha, \alpha+d, \dots, \alpha+5d$. Thus,

$$3\alpha + (k_1 + k_2)d = 3\alpha + (k_3 + k_4 + k_5)d,$$

where $\{k_1, k_2, k_3, k_4, k_5\} = \{1, 2, 3, 4, 5\}$. Therefore if $d \neq 0$, then

$$k_1 + k_2 = k_3 + k_4 + k_5,$$

which is impossible since $k_1 + k_2 + k_3 + k_4 + k_5 = 15$.

2. It follows from Ceva's theorem that $\dfrac{AM.BA_1.CB_1}{MB.A_1C.B_1A} = 1$, i.e. $\dfrac{CB_1}{B_1A} = \dfrac{CA_1}{A_1B}$. Thus, $A_1B_1 \| AB$ and $\sphericalangle A_1B_1C = \sphericalangle BAC$.

We have $\sphericalangle APM = \sphericalangle A_1PC = \dfrac{\overset{\frown}{A_1C}}{2} = \sphericalangle A_1B_1C = \sphericalangle BAC$. Therefore $\triangle PAM \sim \triangle ACM$ and analogously $\triangle PBM \sim \triangle BCM$. It follows from above that $\dfrac{AP}{AC} = \dfrac{AM}{CM} = \dfrac{BM}{CM} = \dfrac{BP}{BC}$, which implies that $\dfrac{AP}{BP} = \dfrac{AC}{BC} = \dfrac{AL}{BL}$, i.e. PL is the inner bisector of $\sphericalangle APB$. Moreover, $\dfrac{AK}{BK} = \dfrac{AC}{BC} = \dfrac{AP}{BP}$, i.e. PK is the outer bisector of $\sphericalangle APB$. Therefore $\sphericalangle LPB = 90^\circ$ implying that P lies on the circle with diameter KL. Note that the point C lies on the same circle, which completes the proof.

3. It suffices to prove the inequality for $r \in \mathbb{Q} \cap (0, 1)$ and then to let $r \to a$. Set $r = \dfrac{p}{q}$, $u = \sqrt[q]{x}$ and $v = \sqrt[q]{y}$.

We have to prove that

$$\frac{u^q - v^q}{1 - (uv)^q} \leq \frac{u^p - v^p}{1 - (uv)^p}$$

for $v \leq u$ and $p < q$. To do this we shall show that

$$\frac{u^{p+1} - v^{p+1}}{1 - (uv)^{p+1}} \leq \frac{u^p - v^p}{1 - (uv)^p},$$

which implies by induction the above inequality. It is easy to check that the latter inequality can be written as

$$\frac{1 - u^{2p+1}}{u^p(1 - u)} \leq \frac{1 - v^{2p+1}}{v^p(1 - v)},$$

i.e.

$$\sum_{j=1}^{p}(u^j + u^{-j}) \leq \sum_{j=1}^{p}(v^j + v^{-j}),$$

which follows by the inequality $t+\dfrac{1}{t}\le z+\dfrac{1}{z}$, where $0<z\le t\le 1$.

4. We shall solve the problem for $A=\{1,2,\dots,t\}$. Let $F_0=F_1=1$, $F_{n+1}=F_n+F_{n-1}$ for $n\ge 1$ be the Fibonacci sequence. We shall prove by induction that if $F_{n-1}<t\le F_n$, $n\ge 2$, then the desired sum equals n.

1. Since for $t=2$ and $t=3$ Peter needs 2 or 3 leva, respectively, the assertion is true for $n=2$ and $n=3$.

2. Suppose that it is true for $n=k$.

3. Choose $t\in(F_k,F_{k+1}]$ and let Peter ask a question set having s elements. If $s\in(F_{k-1},F_k]$ and the answer is "yes" then Peter gives Ivan 2 leva and by the induction hypothesis he needs additional k leva, i.e. in total $k+2$ leva.

If $s\le F_{k-1}$, then $t-s\ge F_k+1-F_{k-1}=F_{k-2}+1$. If Peter receives answer "yes" then he pays 2 leva and he needs additional $k-1$ leva, i.e. in total $k+1$ leva. It remains to notice that if Peter asks a question set with F_{k-1} elements and the answer is "yes" then he needs $2+k-1=k+1$ leva and for answer "no" he needs $1+k-1=k$ leva.

Since $F_{10}=89$ and $F_{11}>89$, the desired number equals 11.

5. Let $a+b=c$. Thus,

$$(c-a)^3+c-a\le a-a^3 \iff 3ca^2-(3c^2+2)a+c^3+c\le 0.$$

If $c>0$ then

$$0\le D=(3c^2+2)^2-12c(c^3+c)=4-3c^4.$$

Hence $c\le\sqrt[4]{\dfrac{4}{3}}$ with equality when a is the double root of the corresponding quadratic equation. Therefore the maximum possible value of $a+b$ equals $\sqrt[4]{\dfrac{4}{3}}$.

6. Suppose that the pair (m,n) satisfies the condition and let (x,y) be the corresponding integer solution.

It follows by Fermat's little theorem that $a^{13}\equiv a \pmod{13}$, $a^{25}\equiv a \pmod{13}$, $a^{37}\equiv a \pmod{13}$ and therefore

$$\begin{aligned} 0 &= (x-m)^{13}-(x-y)^{25}-(y-n)^{37}\equiv x-m-x+y-y+n\\ &\equiv n-m \pmod{13}. \end{aligned}$$

The same reasoning shows that $n-m$ is congruent to 0 modulo 2, 3, 5 and 7. Then $2.3.5.7.13=2730$ divides $n-m$. Since $|n-m|<2005$ it follows that $n=m$. On the other hand, if $n=m$ then $x=y=m$ is a solution of the given equation, i.e. the desired number equals 2006.

7. a) Let $A_4=AA_2\cap BC$ and let the tangent line to k at A_2 meet AB and AC at points X and Y, respectively. Since A_2A_1 is a diameter, we have $XY\parallel BC$, i.e. $\triangle AXY\sim\triangle ABC$.

Since k is an excircle of $\triangle AXY$, it follows from above that $BA_4 = p - c$ and $CA_4 = p - b$ (we use the standard notation for $\triangle ABC$) and therefore $\dfrac{BA_4}{A_4C} = \dfrac{p-c}{p-b}$. The corresponding equalities for the other two vertices and Ceva's theorem imply that the lines AA_2, BB_2 and CC_2 are concurrent.

b) Denote by Z the intersection point of the tangent line to k at A_3 and BC. We have $\sphericalangle A_1A_3A_4 = 90^\circ$ and $ZA_3 = ZA_1$. It follows easily that $ZA_4 = ZA_1$ and therefore

$$CZ = CA_1 + A_1Z = p - c + ZA_1 = BA_4 + ZA_4 = BZ.$$

Hence the desired ratio equals $1:1$.

8. We first show that there exist teams A, B and C, such that A wins over B, B wins over C and C wins over A. Suppose the contrary and take the shortest cycle of $m \geq 4$ teams $A, B, C_1, \dots, C_t$ (i.e. $t \geq 2$), such that A wins over C_1, C_1 wins over C_2, ..., C_t wins over B and B wins over A. Consider the game between C_2 and A. If C_2 is the winner then we have the desired triple and if A is the winner then we have a shorter cycle.

Now we shall use induction on k. The case $k = 3$ was considered above. Suppose that the teams $A_1, A_2, \dots, A_k$ satisfy the condition of the problem for $3 \leq k < n$. There are two cases to be considered.

Case 1. There exists a team $U \notin \{A_1, A_2, \dots, A_k\}$, for which there are two teams A_i and A_j such that A_i wins over U and U wins over A_j. Without loss of generality assume that A_1 wins over U. Let A_ℓ be the team of the least index that loses from U. Then the following $k+1$ teams

$$A_1, \dots, A_{\ell-1}, U, A_\ell, \dots, A_k$$

have the desired property.

Case 2. For any two teams A_i and A_j and any team U either A_i loses from U or A_i and A_j both win over U.

Partition all teams apart from $A_1, A_2, \dots, A_k$ in two sets S and T, such that all teams from S win over $A_1, A_2, \dots, A_k$, and all teams from T lose from $A_1, A_2, \dots, A_k$. It is clear that $S \cap T = \phi$ and non of S and T is the empty set.

Let $U \in S$ and $V \in T$ be such that V wins over U (such a pair exists due to the condition of the problem). Now the following $k+1$ teams

$$U, A_1, \dots, A_{k-1}, V$$

have the desired property.

Team selection test for 47. IMO

1. We shall prove that one can obtain the opposite table by rearranging the rows and columns of the initial table. Denote the columns from left to right and the rows from up to down by $1, 2, \ldots, n$. Denote by a_{ij} the number written in i-th row and j-th column.

Exchanging rows and columns one obtains: $a_{11} = 1$, $a_{12} = -1$, $a_{22} = 1$, $a_{23} = -1$ (when $a_{21} = -1$ the assertion follows by induction using 2×2 and $(n-1) \times (n-2)$ tables), $a_{33} = 1$, $a_{34} = -1$ (if $a_{31} = -1$, the assertion follows by induction using 3×3 and $(n-3) \times (n-3)$ tables) and so on.

It remains to prove the assertion for the table

(A)

1	-1	0	0	...	0	0
0	1	-1	0	...	0	0
0	0	1	-1	...	0	0
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	1	-1
-1	0	0	0	...	0	1

Proceeding in the same way one obtains from the initial table the following one

(B)

-1	1	0	0	...	0	0
0	-1	1	0	...	0	0
0	0	-1	1	...	0	0
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	-1	1
1	0	0	0	...	0	-1

Now applying the same moves for obtaining the table B from the table A but in reverse order one obtains the opposite of the initial table.

2. *First solution.* It suffices to consider the case when P and Q ($\not\equiv 0$) are relatively prime polynomials and the leading coefficient of Q equals 1. We have

$$(1) \qquad x(x+2)(P(x)Q(x+1) - Q(x)P(x+1)) = Q(x)Q(x+1)$$

for infinitely many x, i.e. for every x. Thus the polynomials $Q(x)$ and $Q(x+1)$ divide $x(x+2)Q(x+1)$ and $x(x+2)Q(x)$ respectively.

Therefore $S(x)Q(x) = x(x+2)Q(x+1)$ and $T(X)Q(x+1) = x(x+2)Q(x)$, where S and T are quadratic polynomials with leading coefficients 1. Hence, $S(x)T(x) = x^2(x+2)^2$. There are three cases to be considered.

Case 1. $S(x) = T(x) = x(x+2)$. Then $Q(x+1) = Q(x)$, i.e. $Q \equiv 1$ and the condition of the problem shows that this is impossible.

Case 2. $S(x) = x^2$ and $T(x) = (x+2)^2$. Then $xQ(x) = (x+2)Q(x+1)$. Therefore $Q(1) = 0$ and it follows by induction that $Q(n) = 0$ for all $n \in \mathbb{N}$. Hence $Q \equiv 0$, a contradiction.

Case 3. $S(x) = (x+2)^2$ and $T(x) = x^2$. Then $(x+2)Q(x) = xQ(x+1)$ and therefore x divides $Q(x)$ and $x+2$ divides $Q(x+1)$, i.e. $x+1$ divides $Q(x)$. It follows that $Q(x) = x(x+1)Q_1(x)$, where Q_1 has leading coefficient 1 and $Q_1(x+1) = Q(x)$. We conclude that $Q_1(x) = 1$ and $Q(x) = x(x+1)$. Now plugging $Q(x)$ in (1) gives

$$(x+2)P(x) - xP(x+1) = x+1. \tag{2}$$

Setting $x = 0$ and $x = -1$ we obtain $P(0) = \frac{1}{2}$ and $P(-1) = -\frac{1}{2}$. Therefore $P(x) = \frac{1}{2} + x + x(x+1)P_1(x)$, where P_1 is a polynomial. Now (2) implies that $P_1(x+1) = P(x)$ and therefore P_1 is a constant.

We conclude that if the polynomials P and Q are relatively prime and $a_0 = 1$, then $Q(x) = x(x+1)$ and $P(x) = \frac{1}{2} + x + cx(x+1)$.

Therefore the answer is

$$Q(x) = x(x+1)R(x) \text{ and } P(x) = \left(\frac{1}{2} + x + cx(x+1)\right)R(x),$$

where R is an arbitrary nonzero polynomial and c is a constant.

Second solution. The given identity can be written as

$$\frac{P(x)}{Q(x)} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{P(x+1)}{Q(x+1)} - \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x+2}\right).$$

Hence it follows by induction that

$$\frac{P(x)}{Q(x)} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{P(x+n)}{Q(x+n)} - \frac{1}{2}\left(\frac{1}{x+n} + \frac{1}{x+n+1}\right).$$

Fixing x and letting $n \to \infty$ we see that $\frac{P(x)}{Q(x)} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) = c$, where c is a constant. Now it is easy to conclude that $Q(x) = x(x+1)R(x)$ and $P(x) = \left(\frac{1}{2} + x + cx(x+1)\right)R(x)$.

Remark. One can prove in a similar manner the following assertion: Let P and Q be polynomials with real coefficients such that

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+a)}$$

for infinitely many $x \in \mathbb{R}$, where a is a real number.

Then $a \in \mathbb{Z}$ and $a \neq 0$. Moreover let $n \in \mathbb{N}$,

$$S_n(x) = \frac{1}{0!n} + \frac{x}{1!(n-1)} + \frac{x(x+1)}{2!(n-2)} + \dots + \frac{x(x+1)\dots(x+n-2)}{(n-1)!1},$$

and $T_n(x) = x(x+1)\dots(x+n-1)$. If $a = n$ then

$$Q(x) = T_n(x)R(x) \text{ и } P(x) = (S_n(x) + cT_n(x))R(x),$$

where R is an arbitrary nonzero polynomial and c is a constant. The case $a = -n$ is treated in the same way after substitutions $P_1(x) = -P(1-x)$ and $Q_1(x) = -Q(1-x)$.

3. The angle equality implies that M and N are isogonal conjugate points in $\triangle ABC$. Therefore $\sphericalangle BCM = \sphericalangle ACN$. Denote by small letters the affixes of the corresponding points in the complex plane. We have that

$$\arg \frac{b-a}{m-a} = \arg \frac{n-a}{c-a}$$

and

$$|(m-a)(n-a)(b-c)| = k.$$

Thus,

$$\frac{(b-a)(c-a)}{(m-a)(n-a)} = \frac{AB.BC.CA}{k} =: K.$$

Analogously $(c-b)(a-b) = K(m-b)(n-b)$ and $(a-c)(b-c) = K(m-c)(n-c)$. After substraction we obtain

$$(b-c)(K(m+n) - (K-1)(b+c) - 2a) = 0.$$

Hence

$$m + n = \frac{(K-1)(b+c) + 2a}{K}$$

and analogously $m+n = \dfrac{(K-1)(c+a)+2b}{K}$. Therefore $K = 3$ and $\dfrac{m+n}{2} = \dfrac{a+b+c}{3}$, which completes the proof.

Remark 1. Actually the numbers m and n are the roots of the derivative of the polynomial $(z-a)(z-b)(z-c)$.

Remark 2. Direct verification shows that the following identity holds true

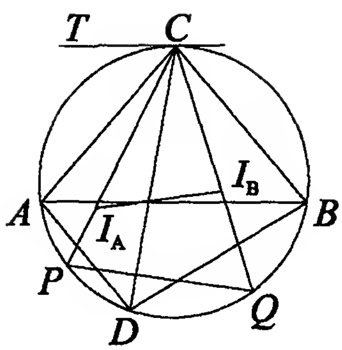
$$\frac{(m-a)(n-a)}{(b-a)(c-a)} + \frac{(m-b)(n-b)}{(c-b)(a-b)} + \frac{(m-c)(n-c)}{(a-c)(b-c)} = 1.$$

It follows by the triangle inequality that if M and N are points in the plane of $\triangle ABC$, then

$$\frac{AM.AN}{AB.AC} + \frac{BM.BN}{BC.BA} + \frac{CM.CN}{CA.CB} \geq 1.$$

The equality occurs if and only if the points M and N are isogonal conjugate.

4. Let $P = CI_A \cap k$ and $Q = CI_B \cap k$. First we prove that the circumcircle k_1 of $\triangle I_AI_BC$ is tangent to k if and only if $I_AI_B || PQ$.



Let T be the point on the tangent line to k at C, for which $\sphericalangle ACT = \sphericalangle ABC$. If k and k_1 are tangent then CT is their common tangent line and therefore

$$\sphericalangle CQP = \sphericalangle TCP = \sphericalangle TCI_A = \sphericalangle CI_BI_A,$$

i.e. $I_AI_B || PQ$. Conversely, if $I_AI_B || PQ$, then $\sphericalangle TCI_A = \sphericalangle TCP = \sphericalangle CQP = \sphericalangle CI_BI_A$, implying that the line CT is tangent to k_1.

It remains to prove that $I_AI_B || PQ \iff \dfrac{AD}{BD} = \dfrac{AC+CD}{BC+CD}$.

Since $PI_A = PA = PD$ and $QI_B = QB = QD$, we have

$$\begin{aligned}
I_AI_B || PQ \quad &\iff \quad \frac{CI_A}{CI_B} = \frac{AP}{BQ} \\
&\iff \quad \frac{\dfrac{AC+CD-AD}{2\cos\dfrac{\sphericalangle ACD}{2}}}{\dfrac{BC+CD-BD}{2\cos\dfrac{\sphericalangle BCD}{2}}} = \frac{2R_{ABC}\sin\dfrac{\sphericalangle ACD}{2}}{2R_{ABC}\sin\dfrac{\sphericalangle BCD}{2}} \\
&\iff \quad \frac{AC+CD-AD}{BC+CD-BD} = \frac{\sin\sphericalangle ACD}{\sin\sphericalangle BCD} \\
&\iff \quad \frac{AC+CD-AD}{BC+CD-BD} = \frac{AD}{BD} \iff \frac{AC+CD}{BC+CD} = \frac{AD}{BD}.
\end{aligned}$$

5. We shall prove the following generalization of the given inequality: if $x, y \geq 1$ and $a, b, c > 0$ then

$$\frac{ab}{xa+yb+2c} + \frac{bc}{xb+yc+2a} + \frac{ca}{xc+ya+2b} \leq \frac{a+b+c}{x+y+2}. \tag{1}$$

The given inequality is obtained for $x = \dfrac{6}{5}$ and $y = \dfrac{8}{5}$.

According to the Cauchy-Schwarz inequality we have that

$$\begin{aligned}
ab\frac{(x+y+2)^2}{xa+yb+2c} &= ab\frac{((x-1)+(y-1)+2+2)^2}{(x-1)a+(y-1)b+(a+c)+(b+c)} \\
&\leq ab\left(\frac{x-1}{a} + \frac{y-1}{b} + \frac{4}{a+c} + \frac{4}{b+c}\right) \\
&= (x-1)b + (y-1)a + \frac{4ab}{a+c} + \frac{4ab}{b+c}.
\end{aligned}$$

Summing up this inequality and the other two similar inequalities we obtain

$$\begin{aligned}
&(x+y+2)^2\left(\frac{ab}{xa+yb+2c}+\frac{bc}{xb+yc+2a}+\frac{ca}{xc+ya+2b}\right)\\
\le\ & (x-1)(a+b+c)+(y-1)(a+b+c)+4(a+b+c)\\
=\ & (x+y+2)(a+b+c),
\end{aligned}$$

which proves (1).

Second solution. The inequality is equivalent to

$$\begin{aligned}
0 \le\ & 30[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2]\\
&+11[ab(b-c)^2+bc(c-a)^2+ca(a-b)^2]\\
&+73[ab(c-a)^2+bc(a-b)^2+ca(b-c)^2].
\end{aligned}$$

6. Consider the set $A' = A \cup \{0\}$ instead of A. Let $B = \{a_1, a_2, \ldots, a_k\}$ be a nonempty subset of A'. Set

$$i + B = \{i + a_1 \ (\mathrm{mod}\ p),\ i + a_2 \ (\mathrm{mod}\ p), \ldots,\ i + a_k \ (\mathrm{mod}\ p)\}.$$

Note that the sums of the elements of the sets $i + B$, $i = 0, 1, \ldots, p-1$, are all distinct. Indeed, if for some s and t the sums are equal then

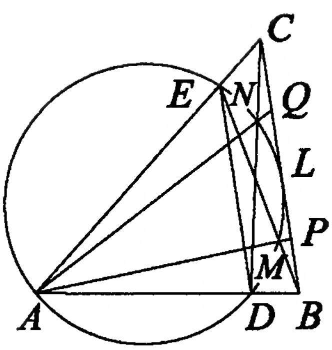
$$\begin{aligned}
&\sum_{i=1}^{k}(s+a_i) \equiv \sum_{i=1}^{k}(t+a_i) \pmod p\\
\Longleftrightarrow\quad & ks + \sum_{i=1}^{k} a_i \equiv kt + \sum_{i=1}^{k} a_i \pmod p\\
\Longleftrightarrow\quad & ks \equiv kt \pmod p
\end{aligned}$$

which is equivalent to $s = t$.

Therefore the set of the subsets of A' (without the empty set and A') partitions into $\frac{2^p-2}{p}$ groups and every group contains p sets. Moreover the sums of the elements of the subsets in every group run over all residues modulo p.

Therefore the number of the subsets having sums divisible by p, equals $\frac{2^p-2}{p}$. Since 0 is included in half of them it follows that the number of the subsets B of A (including the empty set and excluding A) equals $\frac{2^p-2}{2p}$.

Replacing the empty set by A (having sum $\frac{p(p-1)}{2}$, which is divisible by p), we conclude that the answer is $\frac{2^{p-1}-1}{p}$.



7. Since $\measuredangle PBM = \measuredangle MED = \measuredangle BAP$ we have $PB^2 = PM.PA$. Analogously $QC^2 = QN.QA$. Since $BC = 2PQ$ and P lies between B and Q, there exists a point L on PQ such that $PB = PL$ and $QC = QL$. Thus, $PL^2 = PM.PA$, i.e. M lies on the circle k' through A, tangent to BC at L. Analogously $N \in k'$ and therefore $k' = k$. Finally, we obtain that

$$\frac{BL^2}{CL^2} = \frac{BD.BA}{CE.CA} = \frac{BA^2}{CA^2}, \text{ i.e. } \measuredangle BAL = \measuredangle CAL.$$

8. a) Suppose that the inequality $\{a_n a_{n-1} \dots a_1 x\} > \dfrac{1}{a_{n+1}}$ holds for finitely many values of n. Hence there exists s such that for any $n \geq s$ we have $\{a_n a_{n-1} \dots a_1 x\} \leq \dfrac{1}{a_{n+1}}$. Since $\{a_n a_{n-1} \dots a_1 x\}$ is not a rational number (in particular does not equal 0) we obtain that $\{a_n a_{n-1} \dots a_1 x\} < \dfrac{1}{a_{n+1}}$, i.e. $a_{n+1}\{a_n a_{n-1} \dots a_1 x\} < 1$. Using that a_{n+1} is an integer we have

$$\{a_{n+1} a_n a_{n-1} \dots a_1 x\} = \{a_{n+1}\{a_n a_{n-1} \dots a_1 x\}\} = a_{n+1}\{a_n a_{n-1} \dots a_1 x\}.$$

For any $t > s$ we obtain

$$1 > \{a_t a_{t-1} \dots a_s a_{s-1} \dots a_1 x\} = a_t a_{t-1} \dots a_s \{a_{s-1} \dots a_1 x\},$$

a contradiction, since $\lim_{t\to\infty} a_t a_{t-1} \dots a_s = \infty$, but $0 < \{a_{s-1} \dots a_1 x\} < 1$.

b) It is clear that if $a_i = 1$ for some $i > 1$ then $\{a_{i-1} a_{i-2} \dots a_1 x\} > \dfrac{1}{a_i} = 1$ is not true. Suppose that there exists t such that $a_i = 2$ for $i > t$. Then $\{2^p y\} > \dfrac{1}{2}$ for $y = a_t a_{t-1} \dots a_1 x$ and every p. Since $y < 1$ and $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$ we conclude that for every k the inequality $c_k \leq y < c_{k+1}$ holds true, where $c_k = \dfrac{1}{2} + \dfrac{1}{2^2} + \dots + \dfrac{1}{2^k}$. Therefore $2^k y \in \left[2^k c_k, 2^k c_k + \dfrac{1}{2}\right)$, a contradiction to $\{2^k y\} > \dfrac{1}{2}$.

We shall prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence for which $a_i > 1$ for all $i > 1$ and the inequality $a_i > 2$ holds true for infinitely many values of i, then there exist infinitely many $x \in (0, 1)$ such that $x_n > \dfrac{1}{a_{n+1}}$. Set

$$x = \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \frac{b_3}{a_1 a_2 a_3} + \cdots,$$

where $b_1 \le a_i - 1$ and $1 \le b_i \le a_i - 1$ for $i > 1$ and infinitely many of the latter inequalities are strict. Then

$$\begin{aligned} x &= \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \frac{b_3}{a_1 a_2 a_3} + \cdots \\ &< \frac{a_1 - 1}{a_1} + \frac{a_2 - 1}{a_1 a_2} + \frac{a_3 - 1}{a_1 a_2 a_3} + \cdots \\ &= 1 - \frac{1}{a_1} + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2} - \cdots = 1. \end{aligned}$$

The numbers of this type are infinitely many and we have as above that

$$\frac{b_{n+1}}{a_{n+1}} + \frac{b_{n+2}}{a_{n+1} a_{n+2}} + \cdots < 1.$$

Therefore

$$x_n = \frac{b_{n+1}}{a_{n+1}} + \frac{b_{n+2}}{a_{n+1} a_{n+2}} + \cdots > \frac{b_{n+1}}{a_{n+1}} \ge \frac{1}{a_{n+1}}.$$

Remark 1. One can prove that there exist infinitely many irrational numbers that fulfil the condition of b).

Remark 2. One can prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers greater than 1, then every $x \in [0, 1)$ can be written in a unique way in the form $x = \sum_{i=1}^{\infty} \frac{b_i}{a_1 a_2 \dots a_i}$, where b_i are integers such that $0 \le b_i \le a_i - 1$ and infinitely many of the right inequalities are strict.

9. Consider the following three element sets

$$\begin{aligned} N_1 &= \left\{ \frac{p_3 p_4 \cdots p_{n-1}}{p_1}, \frac{p_2 p_3 \cdots p_{n-1}}{p_1}, \frac{p_2 p_3 \cdots p_{n-1} p_n}{p_1} \right\}, \\ N_2 &= \left\{ \frac{p_1 p_4 \cdots p_{n-1}}{p_2}, \frac{p_1 p_3 \cdots p_{n-1}}{p_2}, \frac{p_1 p_3 \cdots p_{n-1} p_n}{p_2} \right\}, \\ &\vdots \\ N_{n-1} &= \left\{ \frac{p_2 p_3 \cdots p_{n-2}}{p_{n-1}}, \frac{p_1 p_2 \cdots p_{n-2}}{p_{n-1}}, \frac{p_1 p_2 \cdots p_{n-2} p_n}{p_{n-1}} \right\}. \end{aligned}$$

It is easy to check that the union of these sets satisfies the condition of the problem and has $3n - 3$ elements.

Suppose that there exists a set having $3n - 2$ elements that satisfies the condition of the problem. It is clear that every prime number appears in denominator at most 3 times.

Let $M_1 \subset M$ be the largest possible set of prime numbers which appear exactly three times in denominator and no two prime numbers from M_1 appear in one and the same denominator. Then the following inequality holds true

$$3|M_1| + 2(n - |M_1|) \ge 3n - 2,$$

i.e. $|M_1| \geq n-2$.

Case 1. Let $|M_1| = n$. Then N is a union of sets of the type $N_i = \left\{\frac{a_i}{p_i}, \frac{b-i}{p_i}, \frac{c_i}{p_i}\right\}$ and it follows that p_i does not appear in at most one of the nominators of the remaining fractions. In every N_i there are at least two fractions whose nominators do not contain one p_j; otherwise there are two equal fractions. Therefore $2n \leq n$, a contradiction.

Case 2. Let $|M_1| = n-1$. There are two cases:

a) The set N is a union of $n-1$ sets of the form $N_i = \left\{\frac{a_i}{xp_i}, \frac{b-i}{p_i}, \frac{c_i}{p_i}\right\}$, where $x = 1$ or $x = p_n$ and $N_n = \left\{\frac{r}{p_n}, \frac{s}{p_n}\right\}$. If a denominator of a fraction not in N_n is divisible by p_n then we have as above that $2(n-1)+1 \leq n$, i.e. $n \leq 1$, a contradiction. Otherwise the denominators of at most three fraction are divisible by p_n. Therefore $2(n-1)+1 \leq n-1+3$ implying $n \leq 3$. It is easy to be seen that $n = 3$ gives no solution.

b) Consider the set $N_n = \left\{\frac{t}{p_n}\right\}$. Then observations analogous to those in a) imply $2(n-1) \leq n-1+5$, i.e. $n \leq 6$. For every $n = 3, 4, 5, 6$ it is easy to find an example of sets with $3n-2$ elements.

Case 3. Let $|M_1| = n-2$. To find a better N one should have $n-2$ sets obtained by the primes from M_1 and two sets with two elements each obtained from p_{n-1} and p_n. As above we obtain $2(n-2)+2 \leq (n-2)+3+3$, implying $n \leq 6$. Hence in this case we do not find a better N.

When $n = 3, 4, 5, 6$ the answer is $3n-2$, and for $n \geq 7$ the answer is $3n-3$.

10. We rewrite the recurrence relation as

$$\frac{(n+2)a_{n-1}}{4a_n a_{n+1}} + \frac{1}{a_n a_{n+1} a_{n+2}} = \frac{(n+3)a_n}{4a_{n+1}a_{n+2}} \iff (n+2)a_{n+2} = \frac{(n+3)a_n^2 - 4}{a_{n-1}}$$

for $n \geq 3$. Setting $n = 2$ in the initial relation we obtain $4(a_1+4) = 5a_1a_2^2$, implying that $a_1|16$ and $5|a_1+4$. Therefore $a_1 = 16$ and $a_2 = 1$ or $a_1 = 1$ and $a_2 = 2$.

Case 1. Let $a_1 = 16$ and $a_2 = 1$. Then $5a_5 = 6a_3^2 - 4$, $a_3a_6 = 18$ and $7a_7 = 2a_5^2 - 1$ for $n = 3$, 4 and 5, respectively. Since $a_3 \equiv \pm 2 \pmod 5$ and a_3 as a divisor of 18 we conclude that $a_3 = 3$ or $a_3 = 18$. The direct check of both values shows no solutions in this case.

Case 2. Let $a_1 = 1$ and $a_2 = 2$. Then $n = 3$ and $n = 4$ give $5a_5 + 2 = 3a_3^2$ and $a_3a_6 = 18$, respectively. Again $a_3 \equiv \pm 2 \pmod 5$ and we see that $a_3 = 3$ and $a_6 = 6$ or $a_3 = 18$ and $a_6 = 1$. In the second case we obtain $a_5 = 194$ which gives a contradiction with $8a_8 = \dfrac{9a_6^2 - 4}{a_5}$.

In the first case $a_5 = 5$ and hence the only possible values are $a_i = i$ for $i = 1, 2, \ldots, 6$. Now easy induction shows that $a_n = n$ for all n.

11. Denote by $D(a,b)$ the set of the divisors of a, which are greater than or equal to b. Thus, $|D(a,b)| = d(a,b)$. Every integer k, $1 \le k \le 4$, belongs to at most one of the sets

$$D(3n+1,1), D(3n+2,2), \dots, D(4n,n). \tag{1}$$

Every integer k, $1 \le k \le n$, $3n+1 \le k \le 4n$ belongs to exactly one of the sets (1). The integers k, $2n+1 \le k \le 3n$ do not appear in the sets (1).

Let $n+1 \le k \le 2n$, i.e. $k = n+i$, $i = 1, \dots, n$. If k belongs to one of the sets (1), then

$$3n+1 \le 2(n+i) \le 4n$$

or

$$3n+1 \le 3(n+i) \le 4n.$$

We conclude that $i = \left\lceil \frac{n+1}{2} \right\rceil, \dots, n$ or $i = 1, \dots, \left\lfloor \frac{n}{3} \right\rfloor$. The number of the integers from the interval $[n+1, 2n]$ that belong to exactly one of the sets (1) equals $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{3} \right\rfloor$. Thus

$$|D(3n+1,1)| + |D(3n+2,2)| + \dots + |D(4n,n)| = 2n + \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{3} \right\rfloor.$$

and therefore $n = 708$.

12. It is easy to observe that there are at most two acute angles in every convex m-gon. Moreover, if there are two acute angles then they are located at one and the same side.

Fix $l = 0, 1, \dots, n-1$ and let A and B be two vertices of M such that there are l vertices on the arc $\overset{\frown}{AB}$. Consider the following expression

$$\binom{l}{m-2} + \binom{n}{m-2} - \binom{n-l-1}{m-2} - (l+1)\binom{n-l-1}{m-3}.$$

We count in it all m-gons having two acute angles to the side AB and all m-gons having one acute angle to the right of AB.

It is easy to see that summation on $l = 0, 1, \dots, n-1$ and then multiplication by $2n+1$ counts every m-gon exactly ones.

Now using the identity $\sum_{s=0}^{k} \binom{s}{t} = \binom{k+1}{t+1}$ we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-1}\left[\binom{l}{m-2} + \binom{n}{m-2} - \binom{n-l-1}{m-2} - (l+1)\binom{n-l-1}{m-3}\right] \\
=\ &\sum_{l=0}^{n-1}\binom{l}{m-2} + n\binom{n}{m-2} - \sum_{s=0}^{n-1}\binom{s}{m-2} - (n-s)\sum_{s=0}^{n-1}\binom{s}{m-3} \\
=\ &n\binom{n}{m-2} - n\sum_{s=0}^{n-1}\binom{s}{m-3} + (s+1)\sum_{s=0}^{n-1}\binom{s}{m-3} - \sum_{s=0}^{n-1}\binom{s}{m-3} \\
=\ &(m-2)\sum_{s=0}^{n-1}\binom{s+1}{m-2} - \binom{n}{m-2} = (m-2)\binom{n+1}{m-1} - \binom{n}{m-2} \\
=\ &\frac{mn-2n-1}{m-1}\binom{n}{m-2}.
\end{aligned}$$

Therefore the answer is $\dfrac{(2n+1)(mn-2n-1)}{m-1}\dbinom{n}{m-2}$.

Classification of the problems

Key. The problems are distributed in four general fields – Algebra and Analysis, Geometry, Number Theory and Combinatorics. The notation consists of three positions: the competition (C), the grade (g), the number (n) of the problem, followed in brackets by the pages where the problem and it solution appear in the book. Here

$C \in \{W, S, R, N, TBMO, TIMO\}$, where

W stands for Winter Mathematical Competition,

S stands for Spring Mathematical Competition,

R stands for Regional Round of the Bulgarian Mathematical Olympiad,

N stands for National Round of the Bulgarian Mathematical Olympiad,

$TBMO$ stands for Team Selection Test for the Balkan Mathematical Olympiad,

$TIMO$ stands for Team Selection Test for the International Mathematical Olympiad;

$g \in \{8, 9, 10, 11, 12, A\}$, where

the number stands for the grade and A means that the problem is the same for all participants;

$n \in \{1, 2, \ldots, 12\}$

is the number of the problem in the corresponding competition.

Some problems can be classified in more fields.

For example, $W - 10 - 3(15, 96)$ is Problem 3 for grade 10 at the Winter Mathematical Competition, it appears on page 15 and its solution starts on page 96.

Algebra and Analysis

2003

$W-9-2(1,48)$, $W-10-1(1,49)$, $W-11-1(2,51)$, $W-12-1(2,53)$, $S-9-1(3,57)$, $S-10-1(3,58)$, $S-11-1(4,60)$, $S-12-1(4,63)$, $S-12-3(5,64)$, $R-A-3(6,65)$, $R-A-5(6,66)$, $N-A-6(8,71)$, $TBMO-A-2(8,72)$, $TBMO-A-3(8,72)$, $TBMO-A-4(8,73)$, $TIMO-A-2(9,75)$

2004

$W-9-1(10,79)$, $W-10-1(10,80)$, $W-11-1(10,81)$, $W-12-1(11,82)$, $S-9-1(12,86)$, $S-10-1(12,87)$, $S-11-1(13,89)$, $S-12-1(13,90)$, $R-9-1(14,93)$, $R-9-3(14,93)$, $R-9-6(14,95)$, $R-10-1(14,95)$, $R-10-4(15,97)$, $R-11-1(15,98)$, $R-12-2(16,101)$, $R-12-6(16,103)$, $TBMO-A-2(18,108)$, $TBMO-A-5(18,109)$, $TIMO-A-1(19,116)$, $TIMO-A-9(19,116)$, $TIMO-A-10(19,117)$, $TIMO-A-11(20,117)$

2005

$W-9-1(21,119)$, $W-10-1(21,120)$, $W-11-1(22,122)$, $W-11-2(22,122)$, $W-12-1(23,124)$, $W-12-3(23,125)$, $S-8-1(24,129)$, $S-9-1(24,130)$, $S-10-1(25,133)$, $S-10-2(25,133)$, $S-11-1(25,134)$, $S-11-2(25,134)$, $S-12-3(26,137)$, $S-12-4(26,137)$, $R-9-1(27,139)$, $R-9-4(27,140)$, $R-10-1(27,141)$, $R-10-4(28,142)$, $R-11-1(28,143)$, $R-11-4(29,145)$, $R-12-2(29,146)$, $N-A-1(31,150)$, $N-A-3(31,151)$, $TBMO-A-1(32,155)$, $TBMO-A-3(32,155)$, $TBMO-A-5(32,156)$, $TIMO-A-3(33,160)$, $TIMO-A-4(33,162)$

2006

$W-9-1(34,165)$, $W-9-2(34,165)$, $W-10-1(34,167)$, $W-11-1(35,168)$, $W-11-3(35,169)$, $W-12-1(35,170)$, $S-9-1(37,174)$, $S-10-1(38,175)$, $S-11-1(38,177)$, $S-11-2(38,177)$, $S-12-1(39,179)$, $S-12-3(39,180)$, $R-9-1(40,182)$, $R-9-4(40,182)$, $R-10-2(40,184)$, $R-10-3(41,184)$, $R-10-4(41,185)$, $R-11-2(41,186)$, $R-11-4(41,186)$, $R-11-6(42,187)$, $R-12-3(42,188)$, $R-12-4(42,188)$, $N-A-2(43,190)$, $TBMO-A-1(44,194)$, $TBMO-A-3(44,194)$, $TBMO-A-5(44,195)$, $TIMO-A-2(45,197)$, $TIMO-A-5(45,200)$, $TIMO-A-8(46,202)$, $TIMO-A-10(46,204)$

Geometry

2003

$W-9-1(1,48)$, $W-10-2(1,50)$, $W-11-2(2,52)$, $W-12-2(2,54)$, $S-8-2(3,56)$, $S-9-2(3,57)$, $S-10-2(4,59)$, $S-11-2(4,61)$, $S-12-2(5,63)$, $R-A-1(6,65)$, $R-A-4(6,66)$, $N-A-2(7,68)$, $TBMO-A-1(8,72)$, $TIMO-A-5(9,77)$

2004

$W-9-2(10,79)$, $W-10-2(10,80)$, $W-11-2(10,81)$, $W-12-2(11,83)$, $S-8-1(12,85)$, $S-9-2(12,87)$, $S-10-2(12,88)$, $S-11-2(13,89)$, $S-12-2(13,91)$, $R-9-2(14,93)$, $R-10-2(14,95)$, $R-10-5(15,98)$, $R-11-2(15,99)$, $R-11-4(15,100)$, $R-12-3(16,101)$, $R-12-4(16,102)$, $N-A-1(17,104)$, $TBMO-A-4(18,108)$, $TBMO-A-6(18,110)$, $TIMO-A-3(19,113)$, $TIMO-A-5(19,114)$, $TIMO-A-7(19,115)$

2005

$W-9-2(21,119)$, $W-10-2(21,120)$, $W-11-3(22,122)$, $W-12-2(23,124)$, $S-8-2(24,129)$, $S-9-2(24,131)$, $S-10-3(25,133)$, $S-11-3(26,135)$, $S-12-1(26,136)$, $S-12-2(26,136)$, $R-9-2(27,139)$, $R-9-5(27,140)$, $R-10-2(27,141)$, $R-10-5(28,142)$, $R-11-2(28,143)$, $R-11-5(29,145)$, $R-12-3(29,147)$, $R-12-4(30,148)$, $N-A-2(31,151)$, $N-A-4(31,152)$, $TBMO-A-2(32,155)$, $TBMO-A-7(32,158)$, $TIMO-A-1(33,160)$, $TIMO-A-5(33,162)$

2006

$W-9-3(34,165)$, $W-10-2(34,167)$, $W-11-2(35,169)$, $W-12-2(36,170)$, $S-8-2(37,173)$, $S-9-2(37,174)$, $S-10-2(38,176)$, $S-11-3(38,177)$, $S-12-2(39,179)$, $R-9-2(40,182)$, $R-9-5(40,183)$, $R-10-1(40,184)$, $R-10-5(41,185)$, $R-11-1(41,185)$, $R-11-5(42,187)$, $R-12-1(42,188)$, $R-12-5(42,189)$, $N-A-5(43,192)$, $N-A-6(43,192)$, $TBMO-A-2(44,194)$, $TBMO-A-7(44,195)$, $TIMO-A-3(45,199)$, $TIMO-A-4(55,200)$, $TIMO-A-7(45,202)$

Number Theory

2003

$W-9-3(1,49)$, $S-8-3(3,56)$, $R-A-6(6,67)$, $N-A-3(7,69)$, $N-A-5(7,70)$, $TIMO-A-6(9,78)$

2004

$W-10-3(10,81)$, $W-11-3(11,82)$, $W-12-3(11,84)$, $S-9-3(12,87)$, $R-9-4(14,94)$, $R-10-4(15,96)$, $R-11-5(16,100)$, $R-12-1(17,101)$, $N-A-2(17,105)$, $N-A-5(17,106)$, $TBMO-A-8(18,111)$, $TIMO-A-2(19,113)$

2005

$W-9-3(21,119)$, $W-10-3(22,121)$, $W-12-4(23,128)$, $S-8-3(24,130)$, $S-9-4(25,132)$, $S-10-4(25,134)$, $S-11-4(26,135)$, $R-9-3(27,139)$, $R-10-3(27,141)$, $R-11-3(29,144)$, $R-12-1(29,146)$, $N-A-6(31,153)$

2006

$W-10-3(34,167)$, $W-12-4(36,171)$, $S-8-1(37,173)$, $S-8-3(37,173)$, $S-9-4(37,175)$, $R-9-3(40,182)$, $N-A-4(43,191)$, $TBMO-A-6(44,195)$, $TIMO-A-6(45,201)$, $TIMO-A-11(46,204)$

Combinatorics

2003

$W-10-3(1,51)$, $W-11-3(2,53)$, $W-12-3(2,54)$, $S-8-1(3,56)$, $S-9-3(3,57)$, $S-10-3(4,59)$, $S-11-3(4,62)$, $R-A-2(6,65)$, $N-A-1(7,68)$, $N-A-4(7,70)$, $TIMO-A-1(9,75)$, $TIMO-A-3(9,76)$, $TIMO-A-4(9,76)$

2004

$W-9-3(10,79)$, $S-8-2(12,85)$, $S-8-3(12,86)$, $S-11-3(13,90)$, $R-9-5(14,94)$, $R-10-6(15,98)$, $R-11-3(15,99)$, $R-11-6(16,100)$, $R-12-5(16,102)$, $N-A-3(17,105)$, $N-A-4(17,105)$, $TBMO-A-1(18,108)$, $TBMO-A-3(18,108)$, $TBMO-A-7(18,110)$, $TIMO-A-4(19,114)$, $TIMO-A-6(19,115)$, $TIMO-A-8(19,116)$, $TIMO-A-12(20,118)$

2005

$W-9-4(21,120)$, $W-10-4(22,121)$, $W-11-4(22,123)$, $S-8-4(24,130)$, $S-9-3(24,131)$, $R-9-6(27,140)$, $R-10-6(28,142)$, $R-11-6(29,145)$, $R-12-5(30,148)$, $R-12-6(30,148)$, $N-A-5(31,153)$, $TBMO-A-4(32,156)$, $TBMO-A-6(32,157)$, $TBMO-A-8(32,158)$, $TIMO-A-2(33,160)$, $TIMO-A-6(33,163)$

2006

$W-9-4(34,166)$, $W-10-4(35,168)$, $W-11-4(35,169)$, $W-12-3(38,171)$, $S-8-4(37,173)$, $S-9-3(37,175)$, $S-10-3(38,176)$, $S-11-4(39,178)$, $R-9-6(40,183)$, $R-11-3(41,186)$, $N-A-1(43,190)$, $N-A-3(43,191)$, $TBMO-A-4(44,195)$, $TBMO-A-8(44,196)$, $TIMO-A-1(45,197)$, $TIMO-A-9(46,203)$, $TIMO-A-12(46,205)$

List of some standard notations

$\mathbb{N}$ – the set of all natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
$\mathbb{Z}$ – the set of all integer numbers, i.e. $\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$
$\mathbb{Q}$ – the set of all rational numbers, i.e. $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}\}$
$\mathbb{R}$ – the set of all real numbers
$\mathbb{C}$ – the set of all complex numbers, i.e. $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}, i^2 = -1\}$
S_{XYZ} – the area of triangle XYZ
p – semi perimeter of given triangle
$[x]$ – the integer part of x, i.e. the largest integer that does not exceed x
$\{x\}$ – the fractional part of x, i.e. $\{x\} = x - [x]$
(a, b) – the greatest common divisor of integers a and b
$[a, b]$ – the least common multiple of integers a and b

GIL

This book contains problems from all selection tests for BMO and IMO. Most of the problems are regarded as difficult IMO type problems.

The book is intended for undergratuates, high school students and teachers who are interested in olympiad mathematics.

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